Centroaffine minimal surfaces

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Introduction

Centroaffine surfaces

Fundamental examples of centroaffine minimal surfaces

Examples with non-vanishing Tchebychev operator
Centroaffine minimal hypersurfaces:

- Hypersurfaces in the Euclidean space
- Objects in centroaffine differential geometry

\[\text{Study properties of submanifolds which are invariant under the affine transformations fixing the origin.} \]

(centroaffine transformations)

- Defined for non-degenerate centroaffine hypersurfaces
- Extremals for the area integral of the centroaffine metric
Background

- 1994 C. P. Wang: Definition of centroaffine minimal hypersurfaces
- Proper affine hyperspheres centered at the origin are centroaffine minimal.
- Only a few essentially new examples are known even if the case of surfaces.
- The integrability conditions for centroaffine minimal surfaces include Tzitzéica equation:

\[(\log \psi)_{xy} = -\psi - \frac{1}{\psi^2}\]

- 2000 W. Schief: A generalization and a discretization of Tzitzéica transformation for proper affine spheres
Review of Euclidean differential geometry

Euclidean differential geometry: Study properties of submanifolds in $\mathbb{R}^n$ which are invariant under the Euclidean motions.

In the following, we consider surfaces in $\mathbb{R}^3$.

**Gauss formula**

$f : D \to \mathbb{R}^3$: a surface

$(x_1, x_2)$: local coordinates

$\langle \ , \ \rangle$: the standard inner product on $\mathbb{R}^3$

$n$: the unit normal vector field

$\implies$ Gauss formula:

$$f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + \langle f_{x_i x_j}, n \rangle n \quad (i, j = 1, 2)$$ (1)
Definition of centroaffine surfaces

**Definition**

\( f : D \rightarrow \mathbb{R}^3 \): a surface

\( f \): a centroaffine surface

\( \updownarrow \text{ def} \)

\( f \): transversal to the tangent plane

**Gauss formula**

\( f : D \rightarrow \mathbb{R}^3 \): a centroaffine surface

\( (x_1, x_2) \): local coordinates

\[
 f_{x_i x_j} = \tilde{\Gamma}^1_{ij} f_{x_1} + \tilde{\Gamma}^2_{ij} f_{x_2} - h(\partial_{x_i}, \partial_{x_j})f \quad (i, j = 1, 2) \quad (2)
\]
Centroaffine metric

The symmetric $(0, 2)$-tensor field $h$ in Gauss formula (2) is called the centroaffine metric.

**Definition**

$f : D \rightarrow \mathbb{R}^3$: a centroaffine surface

- $f$: non-degenerate (resp. definite, indefinite)
- $h$: non-degenerate (resp. definite, indefinite)

**Proposition**

$f : D \rightarrow \mathbb{R}^3$: a centroaffine surface

- $f$: definite (resp. indefinite)

$\updownarrow$

The Euclidean Gaussian curvature: positive (resp. negative)
Recall two kinds of Gauss formula:

\[ f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + \langle f_{x_i x_j}, n \rangle n \]  \hspace{1cm} (1)

\[ f_{x_i x_j} = \tilde{\Gamma}^1_{ij} f_{x_1} + \tilde{\Gamma}^2_{ij} f_{x_2} - h(\partial_{x_i}, \partial_{x_j}) f \]  \hspace{1cm} (2)

From (1) and (2)

\[ \langle f_{x_i x_j}, n \rangle = -h(\partial_{x_i}, \partial_{x_j}) \langle f, n \rangle \]
For simplicity, we consider indefinite case.

- $f : D \rightarrow \mathbb{R}^3$: an indefinite centroaffine surface
- $K$: the Euclidean Gaussian curvature $< 0$
- $(x, y)$: asymptotic line coordinates
- $\psi := h(\partial_x, \partial_y)$
- $d$: the signed distance from the origin to the tangent plane
- $\rho := -\frac{1}{4} \log \left( -\frac{K}{d^4} \right)$
- $\alpha := \psi \det \begin{pmatrix} f \\ f_x \\ f_{xx} \end{pmatrix} / \det \begin{pmatrix} f_x \\ f_y \\ f_{xy} \end{pmatrix}$
- $\beta := \psi \det \begin{pmatrix} f \\ f_y \\ f_{yy} \end{pmatrix} / \det \begin{pmatrix} f_y \\ f_x \\ f_{yx} \end{pmatrix}$
Gauss formula in asymptotic line coordinates

Gauss formula

\[
\begin{aligned}
    f_{xx} &= \left( \frac{\psi_x}{\psi} + \rho_x \right) f_x + \frac{\alpha}{\psi} f_y \\
    f_{xy} &= -\psi f + \rho_y f_x + \rho_x f_y \\
    f_{yy} &= \left( \frac{\psi_y}{\psi} + \rho_y \right) f_y + \frac{\beta}{\psi} f_x
\end{aligned}
\]  

(3)

Proof

Use

\[
\Gamma_{12}^1 = -\frac{1}{4} \frac{K_y}{K}, \quad \Gamma_{12}^2 = -\frac{1}{4} \frac{K_x}{K}
\]

e tc.
Proposition

The integrability conditions for Gauss formula (3) are

\[
\begin{align*}
(\log |\psi|)_{xy} &= -\psi - \frac{\alpha \beta}{\psi^2} + \rho_x \rho_y \\
\alpha_y + \rho_x \psi_x &= \rho_{xx} \psi \\
\beta_x + \rho_y \psi_y &= \rho_{yy} \psi
\end{align*}
\]  

(4)

If \( \rho \) is constant and \( \alpha, \beta \neq 0 \), changing the coordinates, if necessary, we obtain Tzitzéica equation:

\[
(\log \psi)_{xy} = -\psi - \frac{1}{\psi^2}
\]
Centroaffine scalar curvature

\( f : D \to \mathbb{R}^3 \): an indefinite centroaffine surface

\((x, y)\): asymptotic line coordinates

\( \kappa \): the scalar curvature of the centroaffine metric \( h \)

(the centroaffine scalar curvature)

\[
\kappa = -\frac{(\log |\psi|)_{xy}}{\psi} \quad (\psi = h(\partial_x, \partial_y))
\]

If \( f \) is flat, i.e., \( \kappa = 0 \), we may assume that \( \psi = 1 \)

\( \implies \) The integrability conditions (4) are equivalent to equation of associativity in topological field theory:

\[
g_{xxx}g_{yyy} - g_{xxy}g_{xyy} + 1 = 0, \quad (5)
\]

where

\[
\rho = g_{xy}, \quad \alpha = g_{xxx}, \quad \beta = g_{yyy}.
\]
Centroaffine Tchebychev vector field and centroaffine Tchebychev operator

\( f : D \to \mathbb{R}^3 \): a non-degenerate centroaffine surface

\( \tilde{\nabla} \): the connection induced by the centroaffine surface \( f \)

(The corresponding Christoffel symbols are \( \tilde{\Gamma}^k_{ij} \) in (2).)

\( \nabla^h \): the Levi-Civita connection for the centroaffine metric \( h \)

\( C := \tilde{\nabla} - \nabla^h \): the difference tensor

\( T := \frac{1}{2} \text{tr}_h C \): the centroaffine Tchebychev vector field

\( h_{ij} := h(\partial_{x_i}, \partial_{x_j}), \quad (h^{ij}) := (h_{ij})^{-1}, \quad C^k_{ij} \partial_{x_k} := C(\partial_{x_i}, \partial_{x_j}) \)

\( \text{tr}_h C := h^{ij} C^k_{ij} \partial_{x_k} \)

\( \nabla^h T \): the centroaffine Tchebychev operator
Definition of centroaffine minimal surfaces

Centroaffine minimal surfaces: Extremals for the area integral of the centroaffine metric

\[ f : D \rightarrow \mathbb{R}^3 : \text{an indefinite centroaffine surface} \]

\( (x, y): \text{asymptotic line coordinates} \]

\[ \rho = -\frac{1}{4} \log \left( -\frac{K}{d^4} \right) \]

\( \nabla^h T: \text{the centroaffine Tchebychev operator} \)

**Proposition**

\[ f: \text{centroaffine minimal } \iff \rho_{xy} = 0 \]

\[ \iff \text{tr} \nabla^h T = 0 \]
Quadrics

Ellipsoids centered at the origin
Hyperboloids of one sheet centered at the origin
Hyperboloids of two sheets centered at the origin
\[ \Rightarrow T = 0 \]

Elliptic paraboloids removing the vertex which is the origin
Hyperbolic paraboloids removing the saddle point which is the origin
\[ \Rightarrow T \neq 0, \nabla^h T = 0 \]
Proper affine spheres

Proper affine spheres: Blaschke surfaces whose affine shape operator is a non-zero scalar operator.

The center: The point where the affine normals of proper affine spheres meet.

\( f : D \rightarrow \mathbb{R}^3 \): a non-degenerate centroaffine surface

\[ \rho := -\frac{1}{4} \log \left| \frac{K}{d^4} \right| \]

\( T \): the centroaffine Tchebychev vector field

**Proposition**

\( f \): a proper affine sphere centered at the origin \( \iff \rho \): constant \( \iff T = 0 \)
Flat proper affine spheres

\( f : D \rightarrow \mathbb{R}^3 \): a proper affine sphere centered at the origin.

In the following, we put \( f = (X, Y, Z) \), if necessary.

We consider centroaffine surfaces modulo centroaffine congruence.

**Theorem (cf. M. A. Magid-P. J. Ryan 1990)**

If \( f \) is flat, we have the following:

1: \( XYZ = 1 \) (negative definite)
2: \( (X^2 + Y^2)Z = 1 \) (indefinite)
Proper affine spheres with constant centroaffine scalar curvature

\[ f : D \rightarrow \mathbb{R}^3 \]: a proper affine sphere centered at the origin
\( \kappa \): the centroaffine scalar curvature

**Theorem (cf. U. Simon 1991)**

If \( \kappa \) is constant, then \( \kappa = 0, 1 \).
If \( \kappa = 1 \), we have the following:

1. Ellipsoids centered at the origin (positive definite)
2. Hyperboloid of two sheets centered at the origin (negative definite)
3. \( f = A'(u) + vA(u) \), \( A \) is any \( \mathbb{R}^3 \)-valued function s.t.
   \[
   \det \begin{pmatrix}
       A & & \\
       A' & & \\
       A'' & & 
   \end{pmatrix}
   \]
   is non-zero constant. (indefinite)
Theorem (H. L. Liu-C. P. Wang 1995)

If $\nabla^h T = 0$, except the above examples, we have the following:

In 1~3, $a, b, c \in \mathbb{R}$.

1: $X^a Y^b Z^c = 1$, $abc(a + b + c) \neq 0$

2: $\left\{ \exp \left( -a \tan^{-1} \frac{X}{Y} \right) \right\} (X^2 + Y^2)^b Z^c = 1,$
   
   $c(2b + c)(a^2 + b^2) \neq 0$

3: $Z = -X(a \log X + b \log Y)$, $b(a + b) \neq 0$

4: $Z = \pm X \log X + \frac{Y^2}{X}$

5: $f = (e^x, A_1(x)e^y, A_2(x)e^y)$, $A_1$ and $A_2$ are any linearly independent solutions to the differential equation:
   
   $A'' - A' - a(x)A = 0$ for any function $a = a(x)$.
Solutions to equation of associativity

All the examples 1~5 in Theorem due to Liu-Wang are flat.

Proposition

Solutions to equation of associativity (5) corresponding to indefinite flat centroaffine surfaces with $\nabla^h T = 0$ are one of the following:

1: $g = \frac{\alpha}{6} x^3 + \frac{\beta}{6} y^3 + \frac{c_1}{2} x^2 y + \frac{c_2}{2} xy^2$

+ (any polynomials of $x$ and $y$ with degree $\leq 2$)

$\alpha, \beta \in \mathbb{R} \setminus \{0\}, \ c_1, c_2 \in \mathbb{R}, \ \alpha \beta - c_1 c_2 + 1 = 0$

2: By changing $x$ and $y$, if necessary,

$g = \frac{c_1}{2} x^2 y + \frac{c_2}{2} xy^2 + c_3 xy + (\text{any function of } x)$

+ (any polynomial of $y$ with degree $\leq 2$)

$c_1, c_2, c_3 \in \mathbb{R}, \ c_1 c_2 = 1$
Examples with constant centroaffine scalar curvature

- Centroaffine minimal \(\iff\) \(\text{tr} \nabla^h T = 0\)
- All the above examples: \(\nabla^h T = 0\)
- 2006 F: Classification of centroaffine minimal surfaces with constant \(\kappa\) under the assumption on some cubic differentials
- Obtained new examples. \((\nabla^h T \neq 0)\)
- Indefinite case:

\[
\begin{align*}
    f &= \left( \frac{e^{-u}}{u} \cos \nu, \frac{e^{-u}}{u} \sin \nu, 1 - \frac{1}{u} \right) \\
    \kappa &= 1 \\
    T &\text{ is an eigenvector of } \nabla^h T \text{ (cf. 2004 L. Vrancken)}
\end{align*}
\]
Ruled surface

- 2009 F: Classification of centroaffine minimal surfaces with constant $\kappa$ such that $\nabla^h T$ is not diagonalizable
- 2010 F: Classification of centroaffine minimal surfaces with constant $\kappa$ and constant Pick function
- Pick function:
  \[ J = \frac{1}{2} \| C \|^2 = \frac{1}{2} h_{kr} h^{ip} h^{jq} C_{ij}^k C_{pq} \quad (C: \text{the difference tensor}) \]
- In both classification, obtained new examples with $\kappa = 0, 1$.
- One is a ruled surface:
  \[ f = A'(u) + vA(u), \quad A \text{ is an } \mathbb{R}^3\text{-valued function such that} \]
  \[ \det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0 \]
- $\kappa = 1, J = 0$
Another example is flat. \((\kappa = 0)\)

\[ f = \left( \sum_{n=0}^{\infty} \varphi_{n,1}(x)y^n, \sum_{n=0}^{\infty} \varphi_{n,2}(x)y^n, \sum_{n=0}^{\infty} \varphi_{n,3}(x)y^n \right) \]

\((x, y)\): asymptotic line coordinates around \((x_0, 0)\) such that \(x_0 \neq 0\)

\(\varphi_{0,1}, \varphi_{0,2}, \varphi_{0,3}\): Linearly independent solutions to the differential equation:

\[ x\varphi''' + \varphi'' - \varphi = 0 \]  

\(\varphi_{n+1,i} = \frac{x}{n+1}\varphi_{n,i}''\quad (i = 1, 2, 3)\)

\(J = -1\)
Solution to equation of associativity

Proposition

By changing the coordinates, if necessary, solution to equation of associativity (5) corresponding to the above example is

\[ g = \frac{1}{6}x^3 y - \frac{1}{2}y^2 \log y \]

\[ + \text{(any polynomials of } x \text{ and } y \text{ with degree } \leq 2) \]
Solution to the third order differential equation

The differential equation (6) can be solved by using Meijer G-functions:

\[
\phi = c_1 G^{2,0}_{0,3} \left( \frac{x^2}{8} \left| \frac{1}{2}, \frac{1}{2}, 0 \right. \right) + ic_2 G^{1,0}_{0,3} \left( -\frac{x^2}{8} \left| \frac{1}{2}, \frac{1}{2}, 0 \right. \right)
+ c_3 G^{1,0}_{0,3} \left( -\frac{x^2}{8} \left| 0, \frac{1}{2}, \frac{1}{2} \right. \right) \quad (c_1, c_2, c_3 \in \mathbb{R})
\]

The second and the third terms can be written by using the generalized hypergeometric function \( {}_0F_2 \).
Meijer G-function

- Meijer G-function:

\[
G_{m,n}^{p,q}(z \mid a_1, \ldots, a_p) \bigg| b_1, \ldots, b_q
\]

\[
= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s) \prod_{j=n+1}^{p} \Gamma(a_j + s)} z^{-s} ds
\]

- \( m, n, p, q \in \mathbb{Z}, 0 \leq m \leq q, 0 \leq n \leq p \)
- Satisfies a linear differential equation of order \( \max\{p, q\} \).
Thank you for your attention.