Centroaffine minimal surfaces

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Centroaffine minimal hypersurfaces

Centroaffine minimal hypersurfaces:

- ∘ Hypersurfaces in the Euclidean space
- ∘ Objects in centroaffine differential geometry
  ↑
  Study properties of submanifolds which are invariant under the affine transformations fixing the origin.
  (centroaffine transformations)
- ∘ Defined for non-degenerate centroaffine hypersurfaces
- ∘ Extremals for the area integral of the centroaffine metric

Background

- 1994 C. P. Wang: Definition of centroaffine minimal hypersurfaces
- Proper affine hyperspheres centered at the origin are centroaffine minimal.
- Only a few essentially new examples are known even if the case of surfaces.
- The integrability conditions for centroaffine minimal surfaces include Tzitzéica equation:

$$(\log \psi)_{xy} = -\psi - \frac{1}{\psi^2}$$

- 2000 W. Schief: A generalization and a discretization of Tzitzéica transformation for proper affine spheres

Review of Euclidean differential geometry

Euclidean differential geometry: Study properties of submanifolds in $\mathbf{R}^n$ which are invariant under the Euclidean motions.

In the following, we consider surfaces in $\mathbf{R}^3$.

Gauss formula

$f : D \to \mathbf{R}^3$: a surface

(x_1, x_2): local coordinates

$\langle\ ,\ \rangle$: the standard inner product on $\mathbf{R}^3$

n: the unit normal vector field

$\Longrightarrow$ Gauss formula:

$$f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + \langle f_{x_i x_j}, n \rangle n \quad (i, j = 1, 2) \tag{1}$$

Definition of centroaffine surfaces

Definition

$f : D \to \mathbf{R}^3$: a surface

$$f\text{: a centroaffine surface}$$
$$\Updownarrow \text{ def}$$
$$f\text{: transversal to the tangent plane}$$

Gauss formula

$f : D \to \mathbf{R}^3$: a centroaffine surface

(x_1, x_2): local coordinates

$$f_{x_i x_j} = \tilde{\Gamma}^1_{ij} f_{x_1} + \tilde{\Gamma}^2_{ij} f_{x_2} - h(\partial_{x_i}, \partial_{x_j}) f \quad (i, j = 1, 2) \tag{2}$$

Centroaffine metric

The symmetric $(0,2)$-tensor field h in Gauss formula (2) is called the centroaffine metric.

Definition

$f: D \to \mathbf{R}^3$: a centroaffine surface

$$f\text{: non-degenerate (resp. definite, indefinite)} \overset{\text{def}}{\Updownarrow} h\text{: non-degenerate (resp. definite, indefinite)}$$

Proposition

$f: D \to \mathbf{R}^3$: a centroaffine surface

$$f\text{: definite (resp. indefinite)} \Updownarrow \text{The Euclidean Gaussian curvature: positive (resp. negative)}$$

Proof of Proposition

Proof

Recall two kinds of Gauss formula:

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} + \langle f_{x_i x_j}, n \rangle n \tag{1}$$

$$f_{x_i x_j} = \tilde{\Gamma}_{ij}^1 f_{x_1} + \tilde{\Gamma}_{ij}^2 f_{x_2} - h(\partial_{x_i}, \partial_{x_j}) f \tag{2}$$

From (1) and (2)

$$\langle f_{x_i x_j}, n \rangle = -h(\partial_{x_i}, \partial_{x_j}) \langle f, n \rangle$$

Notations

For simplicity, we consider indefinite case.

$f : D \to \mathbf{R}^3$: an indefinite centroaffine surface

K: the Euclidean Gaussian curvature < 0

(x, y): asymptotic line coordinates

$\psi := h(\partial_x, \partial_y)$

d: the signed distance from the origin to the tangent plane

$$\rho := -\frac{1}{4}\log\left(-\frac{K}{d^4}\right)$$

$$\alpha := \psi \det\begin{pmatrix} f \\ f_x \\ f_{xx} \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_x \\ f_y \end{pmatrix}$$

$$\beta := \psi \det\begin{pmatrix} f \\ f_y \\ f_{yy} \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_y \\ f_x \end{pmatrix}$$

Gauss formula in asymptotic line coordinates

Gauss formula

$$\begin{cases} f_{xx} = \left(\dfrac{\psi_x}{\psi} + \rho_x\right) f_x + \dfrac{\alpha}{\psi} f_y \\ f_{xy} = -\psi f + \rho_y f_x + \rho_x f_y \\ f_{yy} = \left(\dfrac{\psi_y}{\psi} + \rho_y\right) f_y + \dfrac{\beta}{\psi} f_x \end{cases} \tag{3}$$

Proof

Use

$$\Gamma^1_{12} = -\frac{1}{4}\frac{K_y}{K}, \quad \Gamma^2_{12} = -\frac{1}{4}\frac{K_x}{K}$$

etc.

Integrability conditions

Proposition

The integrability conditions for Gauss formula (3) are

$$\begin{cases} (\log|\psi|)_{xy} = -\psi - \dfrac{\alpha\beta}{\psi^2} + \rho_x\rho_y \\ \alpha_y + \rho_x\psi_x = \rho_{xx}\psi \\ \beta_x + \rho_y\psi_y = \rho_{yy}\psi \end{cases} \tag{4}$$

If ρ is constant and $\alpha, \beta \neq 0$, changing the coordinates, if necessary, we obtain Tzitzéica equation:

$$(\log\psi)_{xy} = -\psi - \frac{1}{\psi^2}$$

Centroaffine scalar curvature

$f : D \to \mathbf{R}^3$: an indefinite centroaffine surface
(x, y): asymptotic line coordinates
κ: the scalar curvature of the centroaffine metric h
(the centroaffine scalar curvature)

$$\kappa = -\frac{(\log|\psi|)_{xy}}{\psi} \quad (\psi = h(\partial_x, \partial_y))$$

If f is flat, i.e., $\kappa = 0$, we may assume that $\psi = 1$
$\Longrightarrow$ The integrability conditions (4) are equivalent to equation of associativity in topological field theory:

$$g_{xxx}g_{yyy} - g_{xxy}g_{xyy} + 1 = 0, \tag{5}$$

where

$$\rho = g_{xy}, \ \alpha = g_{xxx}, \ \beta = g_{yyy}.$$

Centroaffine Tchebychev vector field and centroaffine Tchebychev operator

$f : D \to \mathbf{R}^3$: a non-degenerate centroaffine surface
$\tilde{\nabla}$: the connection induced by the centroaffine surface f
(The corresponding Christoffel symbols are $\tilde{\Gamma}_{ij}^k$ in (2).)
∇^h: the Levi-Civita connection for the centroaffine metric h
$C := \tilde{\nabla} - \nabla^h$: the difference tensor
$T := \frac{1}{2}\mathrm{tr}_h C$: the centroaffine Tchebychev vector field

$$h_{ij} := h(\partial_{x_i}, \partial_{x_j}),\ (h^{ij}) := (h_{ij})^{-1},\ C_{ij}^k \partial_{x_k} := C(\partial_{x_i}, \partial_{x_j})$$

$$\mathrm{tr}_h C := h^{ij} C_{ij}^k \partial_{x_k}$$

$\nabla^h T$: the centroaffine Tchebychev operator

Definition of centroaffine minimal surfaces

Centroaffine minimal surfaces: Extremals for the area integral of the centroaffine metric

$f : D \to \mathbf{R}^3$: an indefinite centroaffine surface

(x, y): asymptotic line coordinates

$$\rho = -\frac{1}{4} \log \left(-\frac{K}{d^4} \right)$$

$\nabla^h T$: the centroaffine Tchebychev operator

Proposition

$$f\text{: centroaffine minimal} \iff \rho_{xy} = 0 \iff \operatorname{tr} \nabla^h T = 0$$

Quadrics

Ellipsoids centered at the origin
Hyperboloids of one sheet centered at the origin
Hyperboloids of two sheets centered at the origin
$\Longrightarrow T = 0$
Elliptic paraboloids removing the vertex which is the origin
Hyperbolic paraboloids removing the saddle point which is the origin
$\Longrightarrow T \neq 0, \nabla^h T = 0$

Proper affine spheres

Proper affine spheres: Blaschke surfaces whose affine shape operator is a non-zero scalar operator
The center: The point where the affine normals of proper affine spheres meet
$f : D \to \mathbf{R}^3$: a non-degenerate centroaffine surface

$$\rho := -\frac{1}{4} \log \frac{|K|}{d^4}$$

T: the centroaffine Tchebychev vector field

Proposition

f: a proper affine sphere centered at the origin $\iff \rho$: constant
$\iff T = 0$

Flat proper affine spheres

$f : D \to \mathbf{R}^3$: a proper affine sphere centered at the origin
In the following, we put $f = (X, Y, Z)$, if necessary.
We consider centroaffine surfaces modulo centroaffine congruence.

Theorem (cf. M. A. Magid-P. J. Ryan 1990)

If f is flat, we have the following:
1: $XYZ = 1$ (negative definite)
2: $(X^2 + Y^2)Z = 1$ (indefinite)

Proper affine spheres with constant centroaffine scalar curvature

$f : D \to \mathbf{R}^3$: a proper affine sphere centered at the origin
κ: the centroaffine scalar curvature

Theorem (cf. U. Simon 1991)

If κ is constant, then $\kappa = 0, 1$.
If $\kappa = 1$, we have the following:

1: Ellipsoids centered at the origin (positive definite)
2: Hyperboloid of two sheets centered at the origin (negative definite)
3: $f = A'(u) + vA(u)$, A is any $\mathbf{R}^3$-valued function s.t.

$$\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix}$$ is non-zero constant. (indefinite)

Centroaffine surfaces with vanishing centroaffine Tchebychev operator

Theorem (H. L. Liu-C. P. Wang 1995)

If $\nabla^h T = 0$, except the above examples, we have the following:
In 1$\sim$3, $a, b, c \in \mathbf{R}$.

1: $X^a Y^b Z^c = 1$, $abc(a+b+c) \neq 0$

2: $\left\{\exp\left(-a\tan^{-1}\dfrac{X}{Y}\right)\right\}(X^2+Y^2)^b Z^c = 1$, $c(2b+c)(a^2+b^2) \neq 0$

3: $Z = -X(a\log X + b\log Y)$, $b(a+b) \neq 0$

4: $Z = \pm X\log X + \dfrac{Y^2}{X}$

5: $f = (e^x, A_1(x)e^y, A_2(x)e^y)$, A_1 and A_2 are any linearly independent solutions to the differential equation: $A'' - A' - a(x)A = 0$ for any function $a = a(x)$.

Solutions to equation of associativity

All the examples 1∼5 in Theorem due to Liu-Wang are flat.

Proposition

Solutions to equation of associativity (5) corresponding to indefinite flat centroaffine surfaces with $\nabla^h T = 0$ are one of the following:

1: $g = \frac{\alpha}{6}x^3 + \frac{\beta}{6}y^3 + \frac{c_1}{2}x^2y + \frac{c_2}{2}xy^2$
$+$(any polynomials of x and y with degree ≤ 2)
$\alpha, \beta \in \mathbf{R} \setminus \{0\}$, $c_1, c_2 \in \mathbf{R}$, $\alpha\beta - c_1c_2 + 1 = 0$

2: By changing x and y, if necessary,
$g = \frac{c_1}{2}x^2y + \frac{c_2}{2}xy^2 + c_3xy +$ (any function of x)
$+$(any polynomial of y with degree ≤ 2)
$c_1, c_2, c_3 \in \mathbf{R}$, $c_1c_2 = 1$

Examples with constant centroaffine scalar curvature

- Centroaffine minimal $\Longleftrightarrow \operatorname{tr} \nabla^h T = 0$
- All the above examples: $\nabla^h T = 0$
- 2006 F: Classification of centroaffine minimal surfaces with constant κ under the assumption on some cubic differentials
- Obtained new examples. ($\nabla^h T \neq 0$)
- Indefinite case:

$$f = \left(\frac{e^{-u}}{u} \cos v, \frac{e^{-u}}{u} \sin v, 1 - \frac{1}{u} \right)$$

- $\kappa = 1$
- T is an eigenvector of $\nabla^h T$ (cf. 2004 L. Vrancken)

Ruled surface

- 2009 F: Classification of centroaffine minimal surfaces with constant κ such that $\nabla^h T$ is not diagonalizable
- 2010 F: Classification of centroaffine minimal surfaces with constant κ and constant Pick function
- Pick function:

$$J = \frac{1}{2}\|C\|^2 = \frac{1}{2} h_{kr} h^{ip} h^{jq} C_{ij}^k C_{pq}^r \quad (C\text{: the difference tensor})$$

- In both classification, obtained new examples with $\kappa = 0, 1$.
- One is a ruled suface:

$$f = A'(u) + vA(u), \ A \text{ is an } \mathbf{R}^3\text{-valued function such that}$$

$$\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0$$

- $\kappa = 1, \ J = 0$

Flat example

- Another example is flat. ($\kappa = 0$)
- $f = \left(\sum_{n=0}^{\infty} \varphi_{n,1}(x) y^n, \sum_{n=0}^{\infty} \varphi_{n,2}(x) y^n, \sum_{n=0}^{\infty} \varphi_{n,3}(x) y^n \right)$

  (x, y): asymptotic line coordinates around $(x_0, 0)$ such that $x_0 \neq 0$

  $\varphi_{0,1}, \varphi_{0,2}, \varphi_{0,3}$: Linearly independent solutions to the differential equation:

$$x\varphi''' + \varphi'' - \varphi = 0 \tag{6}$$

$$\varphi_{n+1,i} = \frac{x}{n+1} \varphi''_{n,i} \qquad (i = 1, 2, 3)$$

- $J = -1$

Solution to equation of associativity

Proposition

By changing the coordinates, if necessary, solution to equation of associativity (5) corresponding to the above example is

$$g = \frac{1}{6}x^3 y - \frac{1}{2}y^2 \log y$$

$$+(\text{any polynomials of } x \text{ and } y \text{ with degree} \leq 2)$$

Solution to the third order differential equaiton

The differential equation (6) can be solved by using Meijer G-functions:

$$\varphi = c_1 G^{2,0}_{0,3}\left(\frac{x^2}{8}\,\middle|\,\begin{matrix} - \\ \frac{1}{2}, \frac{1}{2}, 0\end{matrix}\right) + ic_2 G^{1,0}_{0,3}\left(-\frac{x^2}{8}\,\middle|\,\begin{matrix} - \\ \frac{1}{2}, \frac{1}{2}, 0\end{matrix}\right) + c_3 G^{1,0}_{0,3}\left(-\frac{x^2}{8}\,\middle|\,\begin{matrix} - \\ 0, \frac{1}{2}, \frac{1}{2}\end{matrix}\right) \quad (c_1, c_2, c_3 \in \mathbf{R})$$

The second and the third terms can be written by using the generalized hypergeometric function ${}_0F_2$.

Meijer G-function

- Meijer G-function:

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)} z^{-s}ds$$

- $m, n, p, q \in \mathbf{Z}$, $0 \le m \le q$, $0 \le n \le p$
- Satisfies a linear differential equation of order $\max\{p, q\}$.

Thank you for your attention.