Multi-Hamiltonian structures associated with the space of closed equicentroaffine curves

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Contents

1. Introduction

2. Flows of equicentroaffine curves

3. Hamiltonian formalism

4. Multi-Hamiltonian structures
Curve flows

- A curve flow is a 1-parameter family of a curve.
- Geometric quantities vary under curve flows.
- Soliton equations appear for special curve flows.
- In some cases, Hamiltonian formalism can be applied.
- Also, it admits bi-Hamiltonian structure.
An example and the main result

- Curve flows associated with the KdV equation can be formulated as Hamiltonian system.
- 1995 U. Pinkall: Considered the space of closed equicentroaffine curves to be an infinite dimensional symplectic manifold.
- The equicentroaffine curvature evolves according to the KdV equation when the flow is generated by a Hamiltonian function given by the total equicentroaffine curvature.
- 2010 F-T. Kurose: Generalized Pinkall’s result to the case of higher KdV flows.
- The above flows also have bi-Hamiltonian structure.
- The main result: The level sets defined by Hamiltonians for higher KdV flows have multi-Hamiltonian structures.
Equicentroaffine curves

**Definition**

$I$: an interval

$\gamma : I \rightarrow \mathbb{R}^2$: a plane curve

$\gamma$: an equicentroaffine curve
def

Any tangent line does not go through the origin.

$\gamma : I \rightarrow \mathbb{R}^2$: an equicentroaffine curve

Changing the variable, if necessary, may assume that the areal velocity is constant:

$$\det \begin{pmatrix} \gamma & \gamma' \end{pmatrix} = 1.$$  

We say the curve is parametrized by equicentroaffine arclength.
Equicentroaffine curvature

$\gamma : I \rightarrow \mathbb{R}^2$: an equicentroaffine curve

Parametrized by equicentroaffine arclength:

$$\begin{align*}
\det \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} &= 1 \\
\Rightarrow \\
\det \begin{pmatrix} \gamma \\ \gamma'' \end{pmatrix} &= 0 \\
\Rightarrow \\
\gamma'' &= -\det \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix} \gamma \\
&=: -\kappa \gamma
\end{align*}$$

$\kappa$: the equicentroaffine curvature
Equicentroaffine curves with constant curvature

- If the equicentroaffine curvature is 0, the curve is a piece of a line which does not go through the origin.
- Ellipse:
  \[ a, b > 0 \]
  \[ \gamma(s) := \left( a \cos \frac{s}{ab}, b \sin \frac{s}{ab} \right) \quad (s \in [0, 2\pi ab]) \]
  \[ \Rightarrow s: \text{an equicentroaffine arclength parameter}, \quad \kappa = \frac{1}{a^2 b^2} \]
- Hyperbola:
  \[ a, b > 0 \]
  \[ \gamma(s) := \left( a \cosh \frac{s}{ab}, b \sinh \frac{s}{ab} \right) \quad (s \in \mathbb{R}) \]
  \[ \Rightarrow s: \text{an equicentroaffine arclength parameter}, \quad \kappa = -\frac{1}{a^2 b^2} \]
The fundamental theorem of equicentroaffine curves

Equicentroaffine transformations: Equiaffine transformations fixing the origin

\[ x \in \mathbb{R}^2 \mapsto xA \quad (A \in \text{SL}(2, \mathbb{R})) \]

The fundamental theorem of equicentroaffine curves

\( I \): an interval
\[ \kappa : I \rightarrow \mathbb{R} \]
\[ \iff \exists \gamma : I \rightarrow \mathbb{R}^2: \text{an equicentroaffine curve with equicentroaffine curvature } \kappa \text{ parametrized by equicentroaffine arclength unique up to equicentroaffine transformations} \]
Flows of equicentroaffine curves

Consider a flow of an equicentroaffine curve:

\[ \gamma = \gamma(s, t) : I \times J \rightarrow \mathbb{R}^2 \]

\( I, J \): intervals

For each fixed \( t \in J \), \( \gamma(\cdot, t) \) is an equicentroaffine curve parametrized by equicentroaffine arclength.

**Proposition**

\[ \exists \alpha : I \times J \rightarrow \mathbb{R} \text{ s.t.} \]

\[
\begin{cases}
\begin{pmatrix}
\gamma \\
\gamma_s
\end{pmatrix}_s =
\begin{pmatrix}
0 & 1 \\
-\kappa & 0
\end{pmatrix}
\begin{pmatrix}
\gamma \\
\gamma_s
\end{pmatrix} \\
\begin{pmatrix}
\gamma \\
\gamma_s
\end{pmatrix}_t =
\begin{pmatrix}
-\frac{1}{2} \alpha_s & \alpha \\
-\frac{1}{2} \alpha_{ss} - \kappa \alpha & \frac{1}{2} \alpha_s
\end{pmatrix}
\begin{pmatrix}
\gamma \\
\gamma_s
\end{pmatrix}
\end{cases}
\]

\begin{align}
\star
\end{align}
Proof of Proposition

Proof

Put

\[ \gamma_t = \beta \gamma + \alpha \gamma_s. \]

Then we have

\[ \gamma_{ts} = (\beta_s - \kappa \alpha) \gamma + (\beta + \alpha_s) \gamma_s. \]

Since \( \gamma(\cdot, t) \) is parametrized by equicentroaffine arc length,

\[
0 = \det \begin{pmatrix} \gamma_t \\ \gamma_s \end{pmatrix} + \det \begin{pmatrix} \gamma \\ \gamma_{st} \end{pmatrix} \\
= \det \begin{pmatrix} \beta \gamma + \alpha \gamma_s \\ \gamma_s \end{pmatrix} + \det \begin{pmatrix} \gamma \\ (\beta_s - \kappa \alpha) \gamma + (\beta + \alpha_s) \gamma_s \end{pmatrix} \\
= 2\beta + \alpha_s.
\]
The integrability condition for the system of linear partial differential equations (\(\ast\)) is

\[
\kappa_t = \frac{1}{2} \alpha_{sss} + 2 \kappa \alpha_s + \kappa_s \alpha.
\]

\[
\downarrow
\]

\[
\kappa_t = \Omega \alpha_s, \quad \Omega = \frac{1}{2} D_s^2 + 2 \kappa + \kappa_s D_s^{-1}
\]

\(\Omega\) is the recursion operator of the KdV equation:

\[
\kappa_t = \frac{1}{2} \kappa_{sss} + 3 \kappa \kappa_s.
\]

In particular, when

\[
\alpha = D_s^{-1} \Omega^{n-1} \kappa_s \quad (n \in \mathbb{N}),
\]

we have the \(n\)th KdV equation.
The space of closed equicentroaffine curves

\[ \mathcal{M} : \text{The space of closed equicentroaffine curves parametrized by equicentroaffine arclength with enclosing area } \pi \]

\[ \mathcal{M} = \left\{ \gamma : S^1 \rightarrow \mathbb{R}^2 \left| \det \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix} = 1 \right\} \right. \quad (S^1 = \mathbb{R}/2\pi\mathbb{Z}) \]

\[ \gamma \in \mathcal{M} \]

From the system of linear partial differential equations (*)

\[ T_\gamma \mathcal{M} = \left\{ -\frac{1}{2} \alpha_s \gamma + \alpha \gamma_s \left| \alpha : S^1 \rightarrow \mathbb{R} \right\} \right. \]

\[ X, Y \in T_\gamma \mathcal{M} \]

\[ \omega_0(X, Y) := \int_{S^1} \det \begin{pmatrix} X \\ Y \end{pmatrix} \, ds \]
A presymplectic form

**Proposition**

$\omega_0$ defines a presymplectic form on $\mathcal{M}$.

$\gamma(\cdot, t_1, t_2, t_3) \in \mathcal{M}$: a 3-parameter family of an element of $\mathcal{M}$

$\omega_0$: closed

\[
\frac{\partial}{\partial t_1} \omega_0(\gamma_{t_2}, \gamma_{t_3}) + \frac{\partial}{\partial t_2} \omega_0(\gamma_{t_3}, \gamma_{t_1}) + \frac{\partial}{\partial t_3} \omega_0(\gamma_{t_1}, \gamma_{t_2}) = 0
\]

$X = -\frac{1}{2} \alpha_s \gamma + \alpha \gamma_s$, $Y = -\frac{1}{2} \beta_s \gamma + \beta \gamma_s$ \quad ($\alpha, \beta : S^1 \to \mathbb{R}$)

\[
\omega_0(X, Y) = \int_{S^1} \alpha \beta_s ds
\]
Closedness

\( \gamma(\cdot, t_1, t_2, t_3) \in \mathcal{M} \): a 3-parameter family of an element of \( \mathcal{M} \)

Put

\[ \gamma_{t_i} = -\frac{1}{2} \alpha_{is} \gamma + \alpha_i \gamma_s \quad (\alpha_i : S^1 \to \mathbb{R}, \ i = 1, 2, 3) \]

Since \( \gamma_{t_i t_j} = \gamma_{t_j t_i} \ (i, j = 1, 2, 3) \), we have

\[ \alpha_{it_j} - \alpha_{jt_i} = \alpha_j \alpha_{is} - \alpha_i \alpha_{js}. \]

On the other hand,

\[
\frac{\partial}{\partial t_1} \omega_0(\gamma_{t_2}, \gamma_{t_3}) = \frac{\partial}{\partial t_1} \int_{S^1} \alpha_2 \alpha_3 s \, ds
\]

\[ = \int_{S^1} \alpha_{2t_1} \alpha_3 s \, ds + \int_{S^1} \alpha_2 \alpha_{3st_1} \, ds \]

\[ = \int_{S^1} \alpha_{2t_1} \alpha_3 s \, ds - \int_{S^1} \alpha_2 \alpha_{3t_1} \, ds, \]

which vanishes by the cyclic sum.
Hamiltonian functions

The $n$th KdV equation:

$$\kappa_t = \Omega^n \kappa_s \quad (\kappa : S^1 \to \mathbb{R})$$

has infinite numbers of conserved quantities $\{H_m\}_{m \in \mathbb{N}}$ which can be represented as

$$H_m = \int_{S^1} h_m(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds.$$ 

For example,

$$h_1 = \kappa, \quad h_2 = \frac{1}{2} \kappa^2, \quad h_3 = \frac{1}{2} \kappa^3 - \frac{1}{4} \kappa_s^2.$$
Hamiltonian formalism


$n \in \mathbb{N}$: fixed

\[ X_n := -\frac{1}{2}(\Omega^{n-1}\kappa_s)\gamma + (D_s^{-1}\Omega^{n-1}\kappa_s)\gamma_s \quad (\gamma \in \mathcal{M}) \]

\[ \implies X_n \text{ is a Hamiltonian vector field for } H_n \text{ with respect to } \omega_0: \]

\[ dH_n = \omega_0(X_n, \cdot) \]

In particular, $H_n$ is a Hamiltonian function for the $n$th KdV flow:

\[ \gamma_t = X_n. \]
Another presymplectic form

\[ X, Y \in T_{\gamma}M \]

\[ \omega_1(X, Y) := \int_{S^1} \det \left( X \begin{pmatrix} D_s^2 + \kappa \end{pmatrix} Y \right) ds \]

\[ X = -\frac{1}{2} \alpha_s \gamma + \alpha \gamma_s, \quad Y = -\frac{1}{2} \beta_s \gamma + \beta \gamma_s \quad (\alpha, \beta : S^1 \rightarrow \mathbb{R}) \]

\[ \Downarrow \]

\[ \omega_1(X, Y) = \int_{S^1} \alpha \Omega \beta_s ds \]

**Theorem (F-T. Kurose)**

\( \omega_1 \) defines a presymplectic form on \( M \).

\( X_n \) is a Hamiltonian vector field for \( H_{n+1} \) with respect to \( \omega_1 \).
Hamiltonian vector field and Poisson structure

\((M, \omega)\): a symplectic manifold

\(H \in C^\infty(M)\)

\(X_H\): a Hamiltonian vector field

\(\{ \cdot, H \}\): the Poisson structure

**Proposition**

\[ X_H = \{ \cdot, H \} \]

**Proof**

For \(f \in C^\infty(M)\),

\[ X_H f = df(X_H) = \omega(X_f, X_H) = \{ f, H \} \]
Bi-Hamiltonian structure:

- There exist two Poisson structures on a Poisson manifold.
  \[
  \{ \cdot, \}^1, \{ \cdot, \}^2
  \]

- A Hamiltonian vector field can be expressed in two ways.
  \[
  \{ \cdot, H_2 \}^1 = \{ \cdot, H_1 \}^2
  \]

\(\omega_0\) and \(\omega_1\) define bi-Hamiltonian structure on \(M\).

\[
dH_n = \omega_0(X_n, \cdot) = \omega_1(X_{n-1}, \cdot)
\]
Magri’s theorem

Theorem (F. Magri 1978)

\( M \): a manifold with compatible Poisson structures i.e.

\[
\{ , \}_1, \{ , \}_2 : \text{Poisson structures}
\]

\[
\{ , \}_1 + \{ , \}_2 : \text{Poisson structure}
\]

\[
\{ , \}_1 : \text{non-degenerate (induced by a symplectic structure)}
\]

\[ \exists H_1, H_2 \in C^\infty(M) \text{ s.t.} \]

\[
\{ \cdot , H_2 \}_1 = \{ \cdot , H_1 \}_2 
\]

\[ \implies \exists H_i \in C^\infty(M) (i \in \mathbb{N}) \text{ s.t.} \]

\[
\{ \cdot , H_{i+1} \}_1 = \{ \cdot , H_i \}_2 \quad (\forall i \in \mathbb{N}) \quad (1)
\]

\[
\{ H_i , H_j \}_1 = 0, \quad \{ H_i , H_j \}_2 = 0 \quad (\forall i, j \in \mathbb{N}) 
\]

\[
\{ , \}_1, \{ , \}_2 : \text{Poisson structures}
\]

\[
\{ , \}_1 + \{ , \}_2 : \text{Poisson structure}
\]

\[
\{ , \}_1 : \text{non-degenerate (induced by a symplectic structure)}
\]
Involutiveness

By involutiveness (2), \( \{H_i\} \)'s become first integrals.

**Proposition**

(2) can be deduced from (1).

**Proof**

If \( i > j \),

\[
\{ H_i, H_j \}_1 = \{ H_{i-1}, H_j \}_2 \\
= \{ H_{i-1}, H_{j+1} \}_1.
\]

If \( 1 \leq k < i \),

\[
\{ H_i, H_j \}_1 = \{ H_{i-k}, H_{j+k} \}_1 = \{ H_{i-k}, H_{j+k-1} \}_2.
\]

Devide into two cases that \( i - j \) is odd or even.
The level sets of Hamiltonians

\[ m \in \mathbb{N} \]
\[ C_m := (c_1, \ldots, c_m) \in \mathbb{R}^m \]

\[ \mathcal{M}(C_m) := H_1^{-1}(c_1) \cap \cdots \cap H_m^{-1}(c_m) \]

Assume \( \mathcal{M}(C_m) \neq \emptyset \).
\[ \gamma \in \mathcal{M}(C_m) \]

Proposition

\[ T_{\gamma} \mathcal{M}(C_m) = \left\{ -\frac{1}{2} \alpha_s \gamma + \alpha \gamma_s \ \bigg| \begin{array}{c} \alpha : S^1 \to \mathbb{R} \\ \int_{S^1} \kappa \Omega^k \alpha_s ds = 0 \\ (k = 0, 1, 2, \ldots, m - 1) \end{array} \right\} \]
Presymplectic forms on the level sets

Generalize $\omega_0$ and $\omega_1$.

$X, Y \in T_\gamma \mathcal{M}$

$$X = -\frac{1}{2} \alpha_s \gamma + \alpha \gamma_s, \quad Y = -\frac{1}{2} \beta_s \gamma + \beta \gamma_s \quad (\alpha, \beta : S^1 \to \mathbb{R})$$

$k = 0, 1, 2, \ldots$

Assume $\Omega^k \alpha_s$ and $\Omega^k \beta_s$ can be defined.

$$\omega_k(X, Y) := \int_{S^1} \alpha \Omega^k \beta_s ds$$

**Theorem (F-T. Kurose)**

$\omega_0, \omega_1, \ldots, \omega_{m+1}$ define presymplectic forms on $\mathcal{M}(C_m)$.

For each $k = 0, 1, \ldots, m + 1$, $X_n$ is a Hamiltonian vector field for $H_{n+k}$ with respect to $\omega_k$. 
Key lemma

\[ p, q, r = 0, 1, 2, \ldots \]
\[ i = 1, 2, 3 \]
Assume \( \Omega^{p+1}_i \), \( \Omega^{q+1}_i \) and \( \Omega^{r+1}_i \) can be defined.

\[ A(p, q, r) := \int_{S^1} (\Omega^p \alpha_1)(D_s^{-1} \Omega^q \alpha_2)(\Omega^r \alpha_3)ds \]
\[ + \int_{S^1} (\Omega^p \alpha_2)(D_s^{-1} \Omega^q \alpha_3)(\Omega^r \alpha_1)ds \]
\[ + \int_{S^1} (\Omega^p \alpha_3)(D_s^{-1} \Omega^q \alpha_1)(\Omega^r \alpha_2)ds \]

Lemma

\[ A(p, q, r + 1) + A(q, r, p + 1) + A(r, p, q + 1) \]
\[ - A(p + 1, q, r) - A(q + 1, r, p) - A(r + 1, p, q) = 0 \]
Moment maps

$S^1$ acts on $\mathcal{M}$ by parameter shift:

$$\mathcal{M} \ni \gamma \mapsto \gamma(\cdot + \sigma) \quad (\sigma \in S^1),$$

which is symplectic with respect to $\omega_1$. Moreover, the action is Hamiltonian and the moment map is given by $H_1$.

$S^1$ also acts on $\mathcal{M}(C_m)$ by parameter shift, which is symplectic with respect to $\omega_{m+1}$.

**Theorem (F-T. Kurose)**

The moment map $\mu_{m+1}$ for the $S^1$-action on $\mathcal{M}(C_m)$ with respect to $\omega_{m+1}$ is given by

$$\mu_{m+1}(\gamma) \left( \frac{\partial}{\partial \sigma} \right) = H_{m+1}(\gamma) \quad (\gamma \in \mathcal{M}(C_m)).$$
The Miura transformation:

\[ \kappa = \frac{\sqrt{-1}}{2} \hat{\kappa}_s + \frac{1}{4} \hat{\kappa}^2 \]

If \( \hat{\kappa} \) is a solution to the mKdV equation:

\[ \hat{\kappa}_t = \frac{1}{2} \hat{\kappa}_{sss} + \frac{3}{4} \hat{\kappa}^2 \hat{\kappa}_s, \]

\( \kappa \) is a solution to the KdV equation.

- Higher mKdV equations are associated with curve flows in the Euclidean plane.
- The Miura transformation can be defined geometrically as maps between complexification of the set of closed curves.
- Multi-Hamiltonian structures are connected via the geometric Miura transformation.
Thank you for your attention.