Multi-Hamiltonian structures associated with the space of closed equicentroaffine curves

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Contents

Curve flows

- A curve flow is a 1-parameter family of a curve.
- Geometric quantities vary under curve flows.
- Soliton equations appear for special curve flows.
- In some cases, Hamiltonian formalism can be applied.
- Also, it admits bi-Hamiltonian structure.

An example and the main result

- ∘ Curve flows associated with the KdV equation can be formulated as Hamiltonian system.
- ∘ 1995 U. Pinkall: Considered the space of closed equicentroaffine curves to be an infinite dimensional symplectic manifold.
- ∘ The equicentroaffine curvature evolves according to the KdV equation when the flow is generated by a Hamiltonian function given by the total equicentroaffine curvature.
- ∘ 2010 F-T. Kurose: Generalized Pinkall's result to the case of higher KdV flows.
- ∘ The above flows also have bi-Hamiltonian structure.
- ∘ The main result: The level sets defined by Hamiltonians for higher KdV flows have multi-Hamiltonian structures.

Equicentroaffine curves

Definition

I: an interval

$\gamma : I \to \mathbf{R}^2$: a plane curve

γ: an equicentroaffine curve

$\Updownarrow$ def

Any tangent line does not go through the origin.

$\gamma : I \to \mathbf{R}^2$: an equicentroaffine curve

Changing the variable, if necessary, may assume that the areal velocity is constant:

$$\det \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = 1.$$

We say the curve is parametrized by equicentroaffine arclength.

Equicentroaffine curvature

$\gamma : I \to \mathbf{R}^2$: an equicentroaffine curve
Parametrized by equicentroaffine arclength:

$$\det \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = 1$$

$$\Downarrow$$

$$\det \begin{pmatrix} \gamma \\ \gamma'' \end{pmatrix} = 0$$

$$\Downarrow$$

$$\gamma'' = -\det \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix} \gamma =: -\kappa\gamma$$

κ: the equicentroaffine curvature

Equicentroaffine curves with constant curvature

- If the equicentroaffine curvature is 0, the curve is a piece of a line which does not go through the origin.
- Ellipse:
  $a, b > 0$

$$\gamma(s) := \left(a\cos\frac{s}{ab}, b\sin\frac{s}{ab}\right) \quad (s \in [0, 2\pi ab])$$

$\Longrightarrow s$: an equicentroaffine arclength parameter, $\kappa = \dfrac{1}{a^2b^2}$

- Hyperbola:
  $a, b > 0$

$$\gamma(s) := \left(a\cosh\frac{s}{ab}, b\sinh\frac{s}{ab}\right) \quad (s \in \mathbf{R})$$

$\Longrightarrow s$: an equicentroaffine arclength parameter, $\kappa = -\dfrac{1}{a^2b^2}$

The fundamental theorem of equicentroaffine curves

Equicentroaffine transformations: Equiaffine transformations fixing the origin

$$x \in \mathbf{R}^2 \mapsto xA \quad (A \in \mathrm{SL}(2, \mathbf{R}))$$

The fundamental theorem of equicentroaffine curves

I: an interval

$\kappa : I \to \mathbf{R}$

$\Longrightarrow {}^{\exists}\gamma : I \to \mathbf{R}^2$: an equicentroaffine curve with equicentroaffine curvature κ parametrized by equicentroaffine arclength unique up to equicentroaffine transformations

Flows of equicentroaffine curves

Consider a flow of an equicentroaffine curve:

$$\gamma = \gamma(s,t) : I \times J \to \mathbf{R}^2$$

I, J: intervals

For each fixed $t \in J$, $\gamma(\cdot, t)$ is an equicentroaffine curve parametrized by equicentroaffine arclength.

Proposition

${}^{\exists}\alpha : I \times J \to \mathbf{R}$ s.t.

$$\begin{cases} \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix}_s = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix} \\ \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}\alpha_s & \alpha \\ -\frac{1}{2}\alpha_{ss} - \kappa\alpha & \frac{1}{2}\alpha_s \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix} \end{cases} \qquad (*)$$

Proof of Proposition

Proof

Put

$$\gamma_t = \beta\gamma + \alpha\gamma_s.$$

Then we have

$$\gamma_{ts} = (\beta_s - \kappa\alpha)\gamma + (\beta + \alpha_s)\gamma_s.$$

Since $\gamma(\cdot, t)$ is parametrized by equicentroaffine arclength,

$$\begin{aligned} 0 &= \det\begin{pmatrix} \gamma_t \\ \gamma_s \end{pmatrix} + \det\begin{pmatrix} \gamma \\ \gamma_{st} \end{pmatrix} \\ &= \det\begin{pmatrix} \beta\gamma + \alpha\gamma_s \\ \gamma_s \end{pmatrix} + \det\begin{pmatrix} \gamma \\ (\beta_s - \kappa\alpha)\gamma + (\beta + \alpha_s)\gamma_s \end{pmatrix} \\ &= 2\beta + \alpha_s. \end{aligned}$$

The integrability condition

The integrability condition for the system of linear partial differential equations $(*)$ is

$$\kappa_t = \frac{1}{2}\alpha_{sss} + 2\kappa\alpha_s + \kappa_s\alpha.$$

$$\Downarrow$$

$$\kappa_t = \Omega\alpha_s, \quad \Omega = \frac{1}{2}D_s^2 + 2\kappa + \kappa_s D_s^{-1}$$

Ω is the recursion operator of the KdV equation:

$$\kappa_t = \frac{1}{2}\kappa_{sss} + 3\kappa\kappa_s.$$

In particular, when

$$\alpha = D_s^{-1}\Omega^{n-1}\kappa_s \quad (n \in \mathbf{N}),$$

we have the nth KdV equation.

The space of closed equicentroaffine curves

$\mathcal{M}$: The space of closed equicentroaffine curves parametrized by equicentroaffine arclength with enclosing area π

$$\mathcal{M} = \left\{ \gamma : S^1 \to \mathbf{R}^2 \,\middle|\, \det \begin{pmatrix} \gamma \\ \gamma_s \end{pmatrix} = 1 \right\} \quad (S^1 = \mathbf{R}/2\pi\mathbf{Z})$$

$\gamma \in \mathcal{M}$

From the system of linear partial differential equations $(*)$

$$T_\gamma \mathcal{M} = \left\{ -\frac{1}{2}\alpha_s \gamma + \alpha \gamma_s \,\middle|\, \alpha : S^1 \to \mathbf{R} \right\}$$

$X, Y \in T_\gamma \mathcal{M}$

$$\omega_0(X, Y) := \int_{S^1} \det \begin{pmatrix} X \\ Y \end{pmatrix} ds$$

A presymplectic form

Proposition

ω_0 defines a presymplectic form on $\mathcal{M}$.

$\gamma(\cdot, t_1, t_2, t_3) \in \mathcal{M}$: a 3-parameter family of an element of $\mathcal{M}$

$$\omega_0\text{: closed}$$
$$\Updownarrow$$
$$\frac{\partial}{\partial t_1}\omega_0(\gamma_{t_2}, \gamma_{t_3}) + \frac{\partial}{\partial t_2}\omega_0(\gamma_{t_3}, \gamma_{t_1}) + \frac{\partial}{\partial t_3}\omega_0(\gamma_{t_1}, \gamma_{t_2}) = 0$$

$$X = -\frac{1}{2}\alpha_s\gamma + \alpha\gamma_s, \ \ Y = -\frac{1}{2}\beta_s\gamma + \beta\gamma_s \quad (\alpha, \beta : S^1 \to \mathbf{R})$$
$$\Downarrow$$
$$\omega_0(X, Y) = \int_{S^1} \alpha\beta_s ds$$

Closedness

$\gamma(\cdot, t_1, t_2, t_3) \in \mathcal{M}$: a 3-parameter family of an element of $\mathcal{M}$
Put

$$\gamma_{t_i} = -\frac{1}{2}\alpha_{is}\gamma + \alpha_i\gamma_s \quad (\alpha_i : S^1 \to \mathbf{R},\ i = 1, 2, 3)$$

Since $\gamma_{t_i t_j} = \gamma_{t_j t_i}$ $(i, j = 1, 2, 3)$, we have

$$\alpha_{it_j} - \alpha_{jt_i} = \alpha_j\alpha_{is} - \alpha_i\alpha_{js}.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t_1}\omega_0(\gamma_{t_2}, \gamma_{t_3}) &= \frac{\partial}{\partial t_1}\int_{S^1}\alpha_2\alpha_{3s}ds \\ &= \int_{S^1}\alpha_{2t_1}\alpha_{3s}ds + \int_{S^1}\alpha_2\alpha_{3st_1}ds \\ &= \int_{S^1}\alpha_{2t_1}\alpha_{3s}ds - \int_{S^1}\alpha_{2s}\alpha_{3t_1}ds,\end{aligned}$$

which vanishes by the cyclic sum.

Hamiltonian functions

The *n*th KdV equation:

$$\kappa_t = \Omega^n \kappa_s \quad (\kappa : S^1 \to \mathbf{R})$$

has infinite numbers of conserved quantities $\{H_m\}_{m \in \mathbf{N}}$ which can be represented as

$$H_m = \int_{S^1} h_m(\kappa, \kappa_s, \kappa_{ss}, \dots) ds.$$

For example,

$$h_1 = \kappa, \ h_2 = \frac{1}{2}\kappa^2, \ h_3 = \frac{1}{2}\kappa^3 - \frac{1}{4}\kappa_s^2.$$

Hamiltonian formalism

Theorem (cf. U. Pinkall 1995 ($n = 1$), F-T. Kurose 2010)

$n \in \mathbf{N}$: fixed

$$X_n := -\frac{1}{2}(\Omega^{n-1}\kappa_s)\gamma + (D_s^{-1}\Omega^{n-1}\kappa_s)\gamma_s \quad (\gamma \in \mathcal{M})$$

$\Longrightarrow$ X_n is a Hamiltonian vector field for H_n with respect to ω_0:

$$dH_n = \omega_0(X_n, \ \cdot \ )$$

In particular, H_n is a Hamiltonian function for the nth KdV flow:

$$\gamma_t = X_n.$$

Another presymplectic form

$X, Y \in T_\gamma \mathcal{M}$

$$\omega_1(X, Y) := \int_{S^1} \det \begin{pmatrix} X \\ (D_s^2 + \kappa) Y \end{pmatrix} ds$$

$$X = -\frac{1}{2}\alpha_s \gamma + \alpha \gamma_s, \;\; Y = -\frac{1}{2}\beta_s \gamma + \beta \gamma_s \quad (\alpha, \beta : S^1 \to \mathbf{R})$$

$$\Downarrow$$

$$\omega_1(X, Y) = \int_{S^1} \alpha \Omega \beta_s ds$$

Theorem (F-T. Kurose)

ω_1 defines a presymplectic form on $\mathcal{M}$.
X_n is a Hamiltonian vector field for H_{n+1} with respect to ω_1.

Hamiltonian vector field and Poisson structure

(M, ω): a symplectic manifold
$H \in C^\infty(M)$
X_H: a Hamiltonian vector field
$\{\ ,\ \}$: the Poisson structure

Proposition

$$X_H = \{\ \cdot\ , H\}$$

Proof

For $f \in C^\infty(M)$,

$$\begin{aligned} X_H f &= df(X_H) \\ &= \omega(X_f, X_H) \\ &= \{f, H\} \end{aligned}$$

Bi-Hamiltonian structure

Bi-Hamiltonian structure:

- There exist two Poisson structures on a Poisson manifold.

$$\{\ ,\ \}_1,\ \{\ ,\ \}_2$$

- A Hamiltonian vector field can be expressed in two ways.

$$\{\ \cdot\ , H_2\}_1 = \{\ \cdot\ , H_1\}_2$$

ω_0 and ω_1 define bi-Hamiltonian structure on $\mathcal{M}$.

$$dH_n = \omega_0(X_n,\ \cdot\ ) = \omega_1(X_{n-1},\ \cdot\ )$$

Magri's theorem

Theorem (F. Magri 1978)

M: a manifold with compatible Poisson structures i.e.

$\{\ ,\ \}_1$, $\{\ ,\ \}_2$: Poisson structures

$\{\ ,\ \}_1 + \{\ ,\ \}_2$: Poisson structure

$\{\ ,\ \}_1$: non-degenerate (induced by a symplectic structure)

${}^\exists H_1, H_2 \in C^\infty(M)$ s.t.

$$\{\,\cdot\,, H_2\}_1 = \{\,\cdot\,, H_1\}_2$$

$\Longrightarrow$ ${}^\exists H_i \in C^\infty(M)$ $(i \in \mathbf{N})$ s.t.

$$\{\,\cdot\,, H_{i+1}\}_1 = \{\,\cdot\,, H_i\}_2 \quad ({}^\forall i \in \mathbf{N}) \tag{1}$$

$$\{H_i, H_j\}_1 = 0,\ \{H_i, H_j\}_2 = 0 \quad ({}^\forall i, j \in \mathbf{N}) \tag{2}$$

Involutiveness

By involutiveness (2), $\{H_i\}$'s become first integrals.

Proposition

(2) can be deduced from (1).

Proof

If $i > j$,

$$\begin{aligned}\{H_i, H_j\}_1 &= \{H_{i-1}, H_j\}_2 \\ &= \{H_{i-1}, H_{j+1}\}_1.\end{aligned}$$

If $1 \leq k < i$,

$$\{H_i, H_j\}_1 = \{H_{i-k}, H_{j+k}\}_1 = \{H_{i-k}, H_{j+k-1}\}_2.$$

Devide into two cases that $i - j$ is odd or even.

The level sets of Hamiltonians

$m \in \mathbf{N}$
$C_m := (c_1, \ldots, c_m) \in \mathbf{R}^m$

$$\mathcal{M}(C_m) := H_1^{-1}(c_1) \cap \cdots \cap H_m^{-1}(c_m)$$

Assume $\mathcal{M}(C_m) \neq \emptyset$.
$\gamma \in \mathcal{M}(C_m)$

Proposition

$$T_\gamma \mathcal{M}(C_m) = \left\{ -\frac{1}{2}\alpha_s \gamma + \alpha \gamma_s \;\middle|\; \begin{array}{c} \alpha : S^1 \to \mathbf{R} \\ \int_{S^1} \kappa \Omega^k \alpha_s ds = 0 \\ (k = 0, 1, 2, \ldots, m-1) \end{array} \right\}$$

Presymplectic forms on the level sets

Generalize ω_0 and ω_1.
$X, Y \in T_\gamma \mathcal{M}$

$$X = -\frac{1}{2}\alpha_s\gamma + \alpha\gamma_s, \ \ Y = -\frac{1}{2}\beta_s\gamma + \beta\gamma_s \quad (\alpha, \beta : S^1 \to \mathbf{R})$$

$k = 0, 1, 2, \ldots$
Assume $\Omega^k \alpha_s$ and $\Omega^k \beta_s$ can be defined.

$$\omega_k(X, Y) := \int_{S^1} \alpha \Omega^k \beta_s ds$$

Theorem (F-T. Kurose)

$\omega_0, \omega_1, \ldots, \omega_{m+1}$ define presymplectic forms on $\mathcal{M}(C_m)$.
For each $k = 0, 1, \ldots, m+1$, X_n is a Hamiltonian vector field for H_{n+k} with respect to ω_k.

Key lemma

$p, q, r = 0, 1, 2, \dots$
$i = 1, 2, 3$
Assume $\Omega^{p+1}\alpha_{is}$, $\Omega^{q+1}\alpha_{is}$ and $\Omega^{r+1}\alpha_{is}$ can be defined.

$$\begin{aligned} A(p,q,r) := &\int_{S^1} (\Omega^p \alpha_{1s})(D_s^{-1}\Omega^q \alpha_{2s})(\Omega^r \alpha_{3s}) ds \\ &+ \int_{S^1} (\Omega^p \alpha_{2s})(D_s^{-1}\Omega^q \alpha_{3s})(\Omega^r \alpha_{1s}) ds \\ &+ \int_{S^1} (\Omega^p \alpha_{3s})(D_s^{-1}\Omega^q \alpha_{1s})(\Omega^r \alpha_{2s}) ds \end{aligned}$$

Lemma

$$\begin{aligned} &A(p,q,r+1) + A(q,r,p+1) + A(r,p,q+1) \\ &\quad - A(p+1,q,r) - A(q+1,r,p) - A(r+1,p,q) = 0 \end{aligned}$$

Moment maps

S^1 acts on $\mathcal{M}$ by parameter shift:

$$\mathcal{M} \ni \gamma \mapsto \gamma(\cdot + \sigma) \quad (\sigma \in S^1),$$

which is symplectic with respect to ω_1.
Moreover, the action is Hamiltonian and the moment map is given by H_1.

S^1 also acts on $\mathcal{M}(C_m)$ by parameter shift, which is symplectic with respect to ω_{m+1}.

Theorem (F-T. Kurose)

The moment map μ_{m+1} for the S^1-action on $\mathcal{M}(C_m)$ with respect to ω_{m+1} is given by

$$\mu_{m+1}(\gamma)\left(\frac{\partial}{\partial \sigma}\right) = H_{m+1}(\gamma) \quad (\gamma \in \mathcal{M}(C_m)).$$

The Miura transformation

The Miura transformation:

$$\kappa = \frac{\sqrt{-1}}{2}\hat{\kappa}_s + \frac{1}{4}\hat{\kappa}^2$$

If $\hat{\kappa}$ is a solution to the mKdV equation:

$$\hat{\kappa}_t = \frac{1}{2}\hat{\kappa}_{sss} + \frac{3}{4}\hat{\kappa}^2\hat{\kappa}_s,$$

κ is a solution to the KdV equation.

- Higher mKdV equations are associated with curve flows in the Euclidean plane.
- The Miura transformation can be defined geometrically as maps between complexification of the set of closed curves.
- Multi-Hamiltonian structures are connected via the geometric Miura transformation.

Thank you for your attention.