Differential geometry of projective or centroaffine surfaces

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Contents

Overview

$\mathbf{P}^n$: the n-dimensional real projective space

Projective differential geometry: differential geometry of submanifolds in $\mathbf{P}^n$

Many classes of surfaces in projective differential geometry related to integrable systems:

- Projective minimal surfaces
- Isothermally asymptotic surfaces

$\vdots$

Centroaffine minimal surfaces: A class of surfaces in centro-affine differential geometry (1994 C. P. Wang)
Integrable systems (2000 W. K. Schief)
Can be considered as projective surfaces

References

- E. V. Ferapontov, Integrable systems in projective differential geometry, Kyushu J. Math., 54 (2000), 183–215
- C. Rogers and W. K. Schief, Bäcklund and Darboux transformations. Geometry and modern applications in soliton theory. Cambridge Texts in Applied Mathematics, 2002
- T. Sasaki, Projective differential geometry and linear homogeneous differential equations, Rokko Lectures in Mathematics, 5, 1999
- T. Sasaki, Line congruence and transformation of projective surfaces, Kyushu J. Math., 60 (2006), 101–243

⋮

Projective surfaces and surfaces in the Euclidean space

$z : D \to \mathbf{P}^3$: a projective surface
(x, y): local coordinates

$$z(x, y) = [z^1(x, y), z^2(x, y), z^3(x, y), z^4(x, y)]$$

z corresponds to a surface in $\mathbf{R}^3$.
If $z^1 \neq 0$,

$$\hat{z} := \left(\frac{z^2}{z^1}, \frac{z^3}{z^1}, \frac{z^4}{z^1} \right).$$

Symmetric 2-form

$z : D \to \mathbf{P}^3$: a projective surface
(x, y): local coordinates
Assume z_{xy}, z_x, z_y, z are linearly independent on D.

$$\begin{cases} z_{xx} = l z_{xy} + a z_x + b z_y + p z, \\ z_{yy} = m z_{xy} + c z_x + d z_y + q z \end{cases} \tag{1}$$

Define a symmetric 2-form φ by

$$\varphi = l dx^2 + 2 dx dy + m dy^2.$$

Proposition

φ is conformal to the second fundamental form of $\hat{z}$.

Proof of Proposition

Proof

Consider the case

$$z = [\lambda, \lambda f],\ \lambda : D \to \mathbf{R} \setminus \{0\},\ f : D \to \mathbf{R}^3$$

From the first equation of (1)

$$\begin{aligned}(\lambda_{xx}, \lambda_{xx} f + 2\lambda_x f_x + \lambda f_{xx}) \\ = l(\lambda_{xy}, \lambda_{xy} f + \lambda_x f_y + \lambda_y f_x + \lambda f_{xy}) + a(\lambda_x, \lambda_x f + \lambda f_x) \\ + b(\lambda_y, \lambda_y f + \lambda f_y) + p(\lambda, \lambda f)\end{aligned}$$

Hence

$$2\lambda_x f_x + \lambda f_{xx} = l(\lambda_x f_y + \lambda_y f_x + \lambda f_{xy}) + a\lambda f_x + b\lambda f_y$$

Proof of Proposition (continued)

Proof (continued)

Since $\lambda \neq 0$

$$\det \begin{pmatrix} f_{xx} \\ f_x \\ f_y \end{pmatrix} = l \det \begin{pmatrix} f_{xy} \\ f_x \\ f_y \end{pmatrix}$$

Similar computation can be done from the second equation of (1).

On the other hand, multiplying the second fundamental form of $\hat{z}$ by $\|f_x \times f_y\|$, we have

$$\det \begin{pmatrix} f_{xx} \\ f_x \\ f_y \end{pmatrix} dx^2 + 2 \det \begin{pmatrix} f_{xy} \\ f_x \\ f_y \end{pmatrix} dxdy + \det \begin{pmatrix} f_{yy} \\ f_x \\ f_y \end{pmatrix} dy^2.$$

Asymptotic line coordinates

$z : D \to \mathbf{P}^3$: a projective surface
(x, y): local coordinates
z_{xy}, z_x, z_y, z: linearly independent on D
Moreover, we assume z is indefinite, i.e., the symmetric 2-form φ is indefinite.
By the above proposition, we can choose asymptotic line coordinates as (x, y), so that

$$l = m = 0.$$

Put

$$\Phi = \begin{pmatrix} z_{xy} \\ z_x \\ z_y \\ z \end{pmatrix}.$$

A system of linear partial differential equations

Proposition

$$\Phi_x = A\Phi, \ \Phi_y = B\Phi, \tag{2}$$

where

$$A = \begin{pmatrix} a & a_y + bc & b_y + bd + p & bq + p_y \\ 0 & a & b & p \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} d & c_x + ac + q & d_x + bc & cp + q_x \\ 1 & 0 & 0 & 0 \\ 0 & c & d & q \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Integrability condition

Proposition

The integrability condition for the system of linear partial differential equations (2) can be written by

$$\begin{cases} L_y = -2bc_x - cb_x, \\ M_x = -2cb_y - bc_y, \\ bM_y + 2Mb_y + b_{yyy} = cL_x + 2Lc_x + c_{xxx}, \end{cases} \tag{3}$$

where

$$a = \theta_x, \ d = \theta_y, \tag{4}$$

$$\begin{cases} L = \theta_{xx} - \frac{1}{2}\theta_x^2 - b\theta_y - b_y - 2p, \\ M = \theta_{yy} - \frac{1}{2}\theta_y^2 - c\theta_x - c_x - 2q. \end{cases}$$

Proof of Proposition

Proof

The integrability condition is

$$A_y - B_x + [A, B] = O.$$

From the second, third and fourth rows, we have identities.
From the $(1, 1)$-entry,

$$a_y = d_x.$$

Hence $^\exists\theta$ satisfying (4).
From the $(1, 2)$-entry, we have the second equation of (3).
From the $(1, 3)$-entry, we have the first equation of (3).
From the $(1, 4)$-entry and the first and second equations of (3), we have the third equation of (3).

Canonical system

$z : D \to \mathbf{P}^3$: an indefinite projective surface
(x, y): asymptotic line coordinates

$$\begin{cases} z_{xx} = \theta_x z_x + b z_y + p z, \\ z_{yy} = c z_x + \theta_y z_y + q z \end{cases}$$

$\lambda : D \to \mathbf{R} \setminus \{0\}$
If we put $z = \lambda w$,

$$z_x = \lambda_x w + \lambda w_x, \; z_{xx} = \lambda_{xx} w + 2\lambda_x w_x + \lambda w_{xx}.$$

Putting $\lambda = e^{\frac{\theta}{2}}$, we may assume

$$\begin{cases} z_{xx} = b z_y + p z, \\ z_{yy} = c z_x + q z. \end{cases} \tag{5}$$

(canonical system)

Coordinate transformation

For the canonical system (5), consider coordinate transformation:

$$u = f(x),\ v = g(y).$$

$\lambda : D \to \mathbf{R} \setminus \{0\}$
If we put $z = \lambda w$,

$$z_x = \lambda_x w + \lambda w_u f',\ z_{xx} = \lambda_{xx} w + 2\lambda_x w_u f' + \lambda w_{uu} (f')^2 + \lambda w_u f''.$$

If $\lambda = \dfrac{C}{\sqrt{f'g'}}$ $(C \in \mathbf{R} \setminus \{0\})$, we have another canonical system:

$$\begin{cases} w_{uu} = \bar{b} w_v + \bar{p} w, \\ w_{vv} = \bar{c} w_u + \bar{q} w. \end{cases}$$

Transformation rule

Proposition

$$\bar{b} = b\frac{g'}{(f')^2}, \ \bar{p} = \frac{1}{(f')^2}\left(p - \frac{1}{2}b\frac{g''}{g'} + \{f;x\}\right),$$

$$\bar{c} = c\frac{f'}{(g')^2}, \ \bar{q} = \frac{1}{(g')^2}\left(q - \frac{1}{2}c\frac{f''}{f'} + \{g;y\}\right),$$

where $\{f;x\}$ and $\{g;y\}$ are the Schwarzian derivatives of f and g respectively:

$$\{f;x\} := \frac{1}{2}\left(\frac{f''}{f'}\right)' - \frac{1}{4}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2.$$

Schwarzian derivative and linear ordinary differential equation of second order

Proposition

$z_1(x), z_2(x)$: linearly independent $\mathbf{R}$-valued solutions to

$$z_{xx} + \alpha(x) z = 0 \tag{6}$$

$f := \dfrac{z_1}{z_2}$

$\Longrightarrow \alpha = \{f; x\}$

Proof

Differentiating $fz_2 = z_1$ twice and using (6), we have

$$\frac{f''}{f'} = -\frac{2z_2'}{z_2}.$$

Projective metric and Darboux cubic form

The transformation rule for b, c in the canonical system (5):

$$\bar{b} = b\frac{g'}{(f')^2}, \ \bar{c} = c\frac{f'}{(g')^2}$$

Hence

$$\bar{b}\bar{c}dudv = bcdxdy$$

$\Longrightarrow$ $bcdxdy$ is invariant for the projective surface z. (projective metric)

Moreover

$$\bar{b}du^3 + \bar{c}dv^3 = f'g'(bdx^3 + cdy^3)$$

$\Longrightarrow$ The conformal class of $bdx^3 + cdy^3$ is invariant for the projective surface z. (Darboux cubic form)

The case that Darboux cubic form vanishes

Consider the case $b = c = 0$.
The integrability condition (3) becomes

$$p_y = 0, \ q_x = 0.$$

$u_1(x), u_2(x)$: linearly independent **R**-valued solutions to

$$z_{xx} = p(x)z$$

$v_1(y), v_2(y)$: linearly independent **R**-valued solutions to

$$z_{yy} = q(y)z$$

If we put

$$z = [u_1 v_1, u_1 v_2, u_2 v_1, u_2 v_2],$$

z is a projective surface with $b = c = 0$, which is a quadratic:

$$(u_1 v_1)(u_2 v_2) = (u_1 v_2)(u_2 v_1).$$

The case that projective metric vanishes

Consider the case $c = 0$.
The integrability condition (3) becomes

$$(-b_y - 2p)_y = 0, \ (-2q)_x = 0, \ bM_y + 2Mb_y + b_{yyy} = 0.$$

By the transformation rule,

$$\bar{q} = \frac{1}{(g')^2} \left(q(y) + \{g; y\} \right).$$

May assume $q = 0$, so that $M = 0$ and

$$z_{xx} = (\alpha(x)y^2 + \beta(x)y + \gamma(x))z_y + (-\alpha(x)y + \delta(x))z, \ z_{yy} = 0.$$

By a further computation, we have a ruled surface:

$$z = A(x) + yB(x).$$

Definition of projective minimal surfaces

Projective minimal surfaces: Extremals for the integral of the projective metric

$z : D \to \mathbf{P}^3$: an indefinite projective surface

(x, y): asymptotic line coordinates

D: bounded

$\Phi(x, y, t)$: a 1-parameter family of indefinite projective surfaces s.t.

$$\Phi(x, y, 0) = z(x, y), \ \Phi|_{\partial D} = z|_{\partial D}$$

Note that (x, y) are asymptotic line coordinates at $t = 0$.

Review of equiaffine differential geometry

$f : D \to \mathbf{R}^3$: an affine surface with transversal vector field ξ

(x_1, x_2): local coordinates

Gauss formula:

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} + h(\partial_{x_i}, \partial_{x_j})\xi \quad (i, j = 1, 2)$$

h: the affine metric

Two kinds of area elements:

- $\circ\ \theta(\partial_{x_1}, \partial_{x_2}) := \det \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \xi \end{pmatrix}$
- $\circ\ \omega(X_1, X_2) := |\det(h(X_i, X_j))|^{\frac{1}{2}} \quad (\theta(X_1, X_2) = 1)$

Blaschke surfaces

Consider nondegenerate affine surfaces, i.e., the affine metric is nondegenerate, which is independent of choice of transversal vector field.

Proposition

$f : D \to \mathbf{R}^3$: a nondegenerate affine surface
∇: the induced connection
$\Longrightarrow$ Changing the transversal vector field, if necessary, we have

$$\nabla\theta = 0, \ \theta = \omega. \tag{7}$$

Blaschke normal: the transversal vector field satisfying (7)
Blaschke surfaces: affine surfaces with Blaschke normal as transversal vector field
Blaschke metric: the affine metric of Blaschke surfaces

Fubini-Pick invariant

$f : D \to \mathbf{R}^3$: a Blaschke surface
Define a cubic form C by

$$C(X, Y, Z) = X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for vector fields X, Y, Z on the surface f.
(x_1, x_2): local coordinates

$$h_{ij} := h(\partial_{x_i}, \partial_{x_j}), \ (h^{ij}) := (h_{ij})^{-1}$$

$$C_{ijk} := C(\partial_{x_i}, \partial_{x_j}, \partial_{x_k})$$

Fubini-Pick invariant:

$$\frac{1}{8}\|C\|_h^2 = \frac{1}{8} h^{ip} h^{jq} h^{kr} C_{ijk} C_{pqr}$$

Method of equiaffine differential geometry

$z : D \to \mathbf{P}^3$: an indefinite projective surface
(x, y): local coordinates
$z = [1, f]$
$\Longrightarrow$ f becomes a Blaschke surface s.t.

$$\begin{cases} f_{xx} = af_x + bf_y + lf_{xy}, \\ f_{yy} = cf_x + df_y + mf_{xy}. \end{cases}$$

Proposition

If (x, y) are asymptotic line coordinates,

$$\frac{1}{8}\|C\|_h^2\theta = \pm bcdx \wedge dy.$$

In particular, $\|C\|_h^2\theta$ is invariant for the projective surface z.

Blaschke normal and Blaschke metric

$f : D \to \mathbf{R}^3$: an indefinite Blaschke surface, i.e., the Blaschke metric is indefinite, s.t.

$$\begin{cases} f_{xx} = \bar{a} f_x + \bar{b} f_y + \bar{l} f_{xy}, \\ f_{yy} = \bar{c} f_x + \bar{d} f_y + \bar{m} f_{xy} \end{cases}$$

ξ: the Blaschke normal

$$\xi = \zeta f_x + \eta f_y + \lambda f_{xy}$$

If (x, y) are asymptotic line coordinates,

$$\zeta = \eta = 0, \ \lambda^2 = \pm \frac{1}{|f_x, f_y, f_{xy}|}.$$

h: the Blaschke metric

$$(h_{ij}) := \begin{pmatrix} h(\partial_x, \partial_x) & h(\partial_x, \partial_y) \\ h(\partial_y, \partial_x) & h(\partial_y, \partial_y) \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \bar{l} & 1 \\ 1 & \bar{m} \end{pmatrix}$$

Area elements and cubic form

θ, ω: the area elements

$$\theta = \omega = \pm \frac{\sqrt{1 - \bar{l}\bar{m}}}{\lambda} dx \wedge dy$$

C: the cubic form

$$\begin{aligned} C_{111} &:= C(\partial_x, \partial_x, \partial_x) \\ &= \left(\frac{\bar{l}}{\lambda}\right)_x - 2\left(\bar{a} - \frac{\bar{l}}{\lambda}\zeta\right)\frac{\bar{l}}{\lambda} - 2\left(\bar{b} - \frac{\bar{l}}{\lambda}\eta\right)\frac{1}{\lambda} \end{aligned}$$

$$C_{112} = C_{121} = C_{211} := C(\partial_x, \partial_x, \partial_y), \ \ldots$$

If (x, y) are asymptotic line coordinates,

$$C_{112} = C_{121} = C_{211} = C_{122} = C_{212} = C_{221} = 0.$$

In particular, the above proposition holds.

Canonical system via equiaffine differential geometry

$z : D \to \mathbf{P}^3$: an indefinite projective surface
(x, y): asymptotic line coordinates
Canonical system:

$$\begin{cases} z_{xx} = bz_y + pz, \\ z_{yy} = cz_x + qz \end{cases}$$

$$z = \left[e^{-\frac{\varphi}{2}}, e^{-\frac{\varphi}{2}} f\right]$$

$\Longrightarrow$ f is a Blaschke surface s.t.

$$\begin{cases} f_{xx} = \varphi_x f_x + bf_y, \\ f_{yy} = cf_x + \varphi_y f_y. \end{cases}$$

and

$$p = -\frac{1}{2}\varphi_{xx} + \frac{1}{4}\varphi_x^2 + \frac{1}{2}\varphi_y b, \ \ q = -\frac{1}{2}\varphi_{yy} + \cdots .$$

First variation formula

D: bounded

$\Phi(x, y, t)$: a 1-parameter family of indefinite Blaschke surfaces s.t.

$$\Phi(x, y, t) = f + t(\alpha f_x + \beta f_y + \gamma f_{xy}) + o(t) \quad (t \to 0)$$

$$\alpha, \beta, \gamma : D \to \mathbf{R}, \ \alpha|_{\partial D} = \beta|_{\partial D} = \gamma|_{\partial D} = 0$$

Proposition

$$\left.\frac{d}{dt}\right|_{t=0} \int_D \frac{1}{8} \|C\|_h^2 \theta = \mp \frac{1}{2} \int_D \{(bM_y + 2Mb_y + b_{yyy}) + (cL_x + 2Lc_x + c_{xxx})\} \gamma dxdy,$$

where

$$L = -b_y - 2p, \ M = -c_x - 2q.$$

Affine spheres

$z = \left[e^{-\frac{\varphi}{2}}, e^{-\frac{\varphi}{2}} f \right] : D \to \mathbf{P}^3$: an indefinite projective surface given by canonical system

ξ: the Blaschke normal of f

$$\Longrightarrow \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \lambda \begin{pmatrix} \varphi_{xy} + bc & b_y + \varphi_y b \\ c_x + \varphi_x c & \varphi_{xy} + bc \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

A direct computation shows that

$$b_y + \varphi_y b = c_x + \varphi_x c = 0$$

implies the projective minimality:

$$bM_y + 2Mb_y + b_{yyy} = cL_x + 2Lc_x + c_{xxx} = 0.$$

Proposition

Affine spheres are projective minimal, i.e., if f is an affine sphere, z is projective minimal.

Demoulin surfaces and Godeaux-Rozet surfaces

Assume $b, c \neq 0$.
By the transformation rule, we can define projective invariant quadratic forms Pdx^2, Qdy^2 by

$$P := p + \frac{1}{2}b_y - \frac{1}{2}\frac{c_{xx}}{c} + \frac{1}{4}\frac{c_x^2}{c^2}, \quad Q := q + \frac{1}{2}c_x - \frac{1}{2}\frac{b_{yy}}{b} + \frac{1}{4}\frac{b_y^2}{b^2}.$$

Demoulin surfaces: $P = Q = 0$
Godeaux-Rozet surfaces: $P = 0$ or $Q = 0$

Proposition

$$\begin{aligned}\{\text{Affine spheres}\} &\subset \{\text{Demoulin surfaces}\}\\ &\subset \{\text{Godeaux-Rozet surfaces}\}\\ &\subset \{\text{Projective minimal surfaces}\}\end{aligned}$$

Definition of isothermally asymptotic surfaces

$z : D \to \mathbf{P}^3$: an indefinite projective surface
Assume $b, c \neq 0$.

Isothermally asymptotic surfaces: $\left(\log \dfrac{b}{c}\right)_{xy} = 0$

◦ Changing the coordinates, if necessary, the above condition is equivalent to $b = c$.
◦ Affine spheres are isothermally asymptotic.
◦ The surface z defines 3 families of curves (3-web):
  · asymptotic curves
  · zero curves of the Darboux cubic form (Darboux's curves)

For isothermally asymptotic surfaces, the above 3-web is hexagonal, i.e., locally diffeomorphic to 3 families of parallel lines.

Web curvature

Consider a 3-web given by 1-forms $\omega_1, \omega_2, \omega_3$.
Each curve is a zero curve of one of ω_i's.
Normalization: $\omega_1 + \omega_2 + \omega_3 = 0$
$\Longrightarrow {}^{\exists}\gamma$: a 1-form s.t.

$$\begin{cases} d\omega_1 = \omega_1 \wedge \gamma, \\ d\omega_2 = \omega_2 \wedge \gamma, \\ d\omega_3 = \omega_3 \wedge \gamma \end{cases} \qquad \text{(web structure equations)}$$

γ: the Chern connection form
The web curvature K is defined by

$$d\gamma = K\omega_1 \wedge \omega_2.$$

Proposition

Hexagonality $\Longleftrightarrow$ $K = 0$

Proof of Proposition and the case of the 3-web for projective surfaces

Proof

$$K = 0 \Longrightarrow \gamma = d \log f \quad ({}^{\exists} f : \text{locally})$$

$$\Longrightarrow d(f\omega_i) = 0 \quad (i = 1, 2, 3)$$

$$\Longrightarrow f\omega_i = du_i \quad ({}^{\exists} u_i : \text{locally})$$

$$\Longrightarrow u_1 + u_2 + u_3\text{: constant} \quad (\text{hexagonality})$$

The above 3-web for the projective surface is given by

$$\omega_1 = -b^{\frac{1}{3}} dx,\ \omega_2 = -c^{\frac{1}{3}} dy,\ \omega_3 = b^{\frac{1}{3}} dx + c^{\frac{1}{3}} dy$$

$$\Longrightarrow \gamma = -\frac{1}{3}\left(\frac{c_x}{c} dx + \frac{b_y}{b} dy\right)$$

$$\Longrightarrow K = -\frac{1}{3}(bc)^{-\frac{1}{3}}\left(\log \frac{b}{c}\right)_{xy}$$

Definition of centroaffine surfaces

Centroaffine surfaces: affine surfaces with position vector as transversal vector field

Gauss formula:

$$f_{x_i x_j} = \tilde{\Gamma}^1_{ij} f_{x_1} + \tilde{\Gamma}^2_{ij} f_{x_2} - \tilde{h}(\partial_{x_i}, \partial_{x_j}) f \quad (i, j = 1, 2)$$

$\tilde{h}$: the centroaffine metric

Consider indefinite case.

(x, y): asymptotic line coordinates

$\psi := \tilde{h}(\partial_x, \partial_y)$

K: the Euclidean Gaussian curvature of f

d: the signed distance from the origin to the tangent plane (the Euclidean support function)

$\rho := -\frac{1}{4} \log \left(-\frac{K}{d^4} \right)$

Gauss formula in asymptotic line coordinates

Gauss formula

$$\begin{cases} f_{xx} = \left(\dfrac{\psi_x}{\psi} + \rho_x\right) f_x + \dfrac{\alpha}{\psi} f_y, \\ f_{xy} = -\psi f + \rho_y f_x + \rho_x f_y, \\ f_{yy} = \left(\dfrac{\psi_y}{\psi} + \rho_y\right) f_y + \dfrac{\beta}{\psi} f_x, \end{cases} \tag{8}$$

where

$$\alpha = \psi \det \begin{pmatrix} f \\ f_x \\ f_{xx} \end{pmatrix} \bigg/ \det \begin{pmatrix} f \\ f_x \\ f_y \end{pmatrix},$$

$$\beta = \psi \det \begin{pmatrix} f \\ f_y \\ f_{yy} \end{pmatrix} \bigg/ \det \begin{pmatrix} f \\ f_y \\ f_x \end{pmatrix}.$$

Integrability condition

Proposition

The integrability condition for the Gauss formula (8) can be written by

$$\begin{cases} (\log|\psi|)_{xy} = -\psi - \dfrac{\alpha\beta}{\psi^2} + \rho_x\rho_y, \\ \alpha_y + \rho_x\psi_x = \rho_{xx}\psi, \\ \beta_x + \rho_y\psi_y = \rho_{yy}\psi. \end{cases}$$

If ρ is constant and $\alpha, \beta \neq 0$, changing the coordinates, if necessary, we obtain Tzitzéica equation:

$$(\log\psi)_{xy} = -\psi - \frac{1}{\psi^2}.$$

Definition of centroaffine minimal surfaces

Centroaffine minimal surfaces: Extremals for the area integral of the centroaffine metric

$f : D \to \mathbf{R}^3$: an indefinite centroaffine surface

(x, y): asymptotic line coordinates

$\tilde{\nabla}$: the induced connection

$\nabla^{\tilde{h}}$: the Levi-Civita connection for the centroaffine metric $\tilde{h}$

$\tilde{C} := \tilde{\nabla} - \nabla^{\tilde{h}}$: the difference tensor

$T := \frac{1}{2}\mathrm{tr}_{\tilde{h}}\tilde{C}$: the Tchebychev vector field

$\nabla^{\tilde{h}} T$: the Tchebychev operator

Proposition

$$f\text{: centroaffine minimal} \iff \rho_{xy} = 0$$
$$\iff \mathrm{tr}\,\nabla^{\tilde{h}} T = 0$$

Fundamental examples

Proper affine spheres: Blaschke surfaces whose affine shape operator is a non-zero scalar operator

The center: The point where the Blaschke normals of proper affine spheres meet

Proposition

f: a proper affine sphere centered at the origin $\iff \rho$: constant $\iff T = 0$

- ◦ Ellipsoids, hyperboloids centered at the origin $\implies T = 0$
- ◦ Elliptic paraboloids removing the vertex which is the origin, hyperbolic paraboloids removing the saddle point which is the origin $\implies T \neq 0$, $\nabla^{\tilde{h}} T = 0$
- ◦ In 1995, H. L. Liu and C. P. Wang classified centroaffine minimal surfaces with $\nabla^{\tilde{h}} T = 0$.

Examples with non-vanishing Tchebychev operator

$\circ\ f = \left(\dfrac{e^u}{u} \cos v, \dfrac{e^u}{u} \sin v, 1 + \dfrac{1}{u} \right)$ (2006 F)

- The centroaffine curvature is 1.
- $\alpha = \beta$, ρ: linear w.r.t. u, v
- T: an eigenvector of $\nabla^{\tilde{h}} T$ (cf. 2004 L. Vrancken)
- Projective minimal and isothermally asymptotic

$\circ\ f = A'(u) + vA(u)$ (2009 F)

A: an $\mathbf{R}^3$-valued function s.t. $\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0$

- The centroaffine curvature is 1.
- The Pick invariant vanishes.
- $\nabla^{\tilde{h}} T$ is not diagonalizable.
- Projective minimal

Thank you for your attention.