Tchebychev 作用素が半単純でない 平坦中心アファイン曲面

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２０１５年７月２５日（土）
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Euclidean differential geometry and affine differential geometry

Consider hypersurfaces in the Euclidean space.

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  - Study properties invariant under the Euclidean transformation.
- Affine differential geometry
  - Choose a transversal vector field.
    - Equiaffine differential geometry
      - Take a Blaschke normal vector field.
      - Study properties invariant under the equiaffine transformation.
    - Centroaffine differential geometry
      - Take a position vector field.
      - Study properties invariant under the affine transformation fixing the origin.

Gauss-Weingarten formula

Euclidean differential geometry

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

D: the standard flat connection on $\mathbf{R}^{n+1}$

$X, Y \in \mathfrak{X}(M)$

- Euclidean differential geometry

  n: a unit normal vector field

  $\Longrightarrow$ Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y) n & \text{(Gauss)} \\ D_X n = -f_* S X & \text{(Weingarten)} \end{cases}$$

  ∇: the Levi-Civita connection

  h: the second fundamental form

  S: the shape operator

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

D: the standard flat connection on $\mathbf{R}^{n+1}$

$X, Y \in \mathfrak{X}(M)$

- ∘ Affine differential geometry

  ξ: a transversal vector field

  $\Longrightarrow$ Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi & \text{(Gauss)} \\ D_X \xi = -f_* SX + \tau(X)\xi & \text{(Weingarten)} \end{cases}$$

  f is called an affine hypersurface.

  ∇: the induced connection

  h: the affine fundamental form

  Considered as a metric.

  S: the affine shape operator

  τ: the transversal connection form

Nondegenerate, definite or indefinite affine hypersurfaces

Definition

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface

h: the affine fundamental form

$$f\text{: nondegenerate (resp. definite, indefinite)} \overset{\text{def.}}{\Updownarrow} h\text{: nondegenerate (resp. definite, indefinite)}$$

Proposition

The above definition is independent of the choice of the transversal vector field ξ.

Proof

$\bar{\xi} := \varphi\xi + f_* Z \ (\varphi : M \to \mathbf{R} \setminus \{0\},\ Z \in \mathfrak{X}(M)) \implies \varphi\bar{h} = h$

Definition of Blaschke hypersurfaces

$f: M^n \to \mathbf{R}^{n+1}$: a nondegenerate affine hypersurface
θ: the volume form induced by the immersion f
ω: a parallel volume form with respect to the flat connection D

$$\theta(X_1, \ldots, X_n) := \omega(f_* X_1, \ldots, f_* X_n, \xi) \quad (X_1, \ldots, X_n \in \mathfrak{X}(M))$$

ω_h: the volume form with respect to the affine fundamental form h

$X_1, \ldots, X_n \in \mathfrak{X}(M)$ s.t. $\theta(X_1, \ldots, X_n) = 1$

$$\omega_h(X_1, \ldots, X_n) := |\det(h(X_i, X_j))|^{\frac{1}{2}}$$

Proposition

$^\exists\xi$ (unique up to the sign) s.t. $\tau \equiv 0$ & $\theta = \omega_h$

The above ξ is called a Blaschke normal vector field.
f is called a Blaschke hypersurface.
h is called a Blaschke metric.

Affine hyperspheres

Definition

$f : M^n \to \mathbf{R}^{n+1}$: a Blaschke hypersurface
S: the affine shape operator

$$f\text{: an affine hypersphere} \overset{\text{def.}}{\iff} S\text{: a scalar operator}$$

$S = 0$: improper
$S \neq 0$: proper

Proposition

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersphere
ξ: a Blaschke normal vector field

- f: improper $\iff$ ξ: a constant vector
- f: proper $\iff$ The lines through f with direction ξ meet at one point (the center).

Examples

Example (Quadrics)

- A proper noncentral quadric is an improper affine hypersphere.
- A proper central quadric is a proper affine hypersphere.

Theorem (M. A. Magid-P. J. Ryan 1990)

$f = (X, Y, Z) : M^2 \to \mathbf{R}^3$: an affine sphere

If the curvature of the Blaschke metric is 0, then f is one of the following up to the affine congruence.

- $Z = X^2 + Y^2$ (definite, improper)
- $Z = XY + a(X)$, a: an arbitrary function (indefinite, improper)
- $XYZ = 1$ (definite, proper)
- $(X^2 + Y^2)Z = 1$ (indefinite, proper)

Definition of centroaffine hypersurfaces

Definition

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

f: a centroaffine hypersurface

$\Updownarrow$ def.

The position vector intersects with the tangent space transversally at any point.

$$\xi := -\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$$

Weingarten formula:

$$D_X \xi = -f_* X$$

$$\Updownarrow$$

$$S = \text{the identity}, \ \tau \equiv 0$$

The affine fundamental form h is called a centroaffine metric.

Tchebychev operator

Consider nondegenerate centroaffine hypersurfaces.
$f : M^n \to \mathbf{R}^{n+1}$: a centroaffine hypersurface
∇: the induced connection
∇^h: the Levi-Civita connection for the centroaffine metric h
$C := \nabla - \nabla^h$: the difference tensor
$T := \frac{1}{n}\mathrm{tr}_h C$: the Tchebychev vector field

$$\phi(\mathrm{tr}_h C) = \mathrm{tr} A_\phi \quad ({}^\forall \phi \in \Omega^1(M))$$

A: a $(1,1)$ tensor s.t.

$$(\phi \circ C)(X, Y) = h(A_\phi(X), Y) \quad ({}^\forall X, Y \in \mathfrak{X}(M))$$

The $(1,1)$ tensor $\nabla^h T$ is called a Tchebychev operator.

Examples

$f : M^n \to \mathbf{R}^{n+1}$: a centroaffine hypersurface
C: the difference tensor
T: the Tchebychev vector field
$\nabla^h T$: the Tchebychev operator

Maschke-Pick-Berwald's theorem

$C \equiv 0 \iff f$: a quadric centered at the origin

Proposition

$T \equiv 0 \iff f$: a proper affine hypersphere centered at the origin

1995 H. L. Liu-C. P. Wang: Classified centroaffine surfaces with $\nabla^h T \equiv 0$

Definiteness and the Euclidean Gaussian curvature

Consider nondegerate centroaffine surfaces.

Proposition

$f : M \to \mathbf{R}^3$: a centroaffine surface

f: definite (resp. indefinite)

$$\Updownarrow$$

the Euclidean Gaussian curvature: positive (resp. negative)

Proof

(x_1, x_2): local coordinates
Gauss formula:

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} - h(\partial_{x_i}, \partial_{x_j}) f \quad (i = 1, 2)$$

Take the inner product with the unit normal vector field.

Notations

Consider the indefinite case.

$f : M \to \mathbf{R}^3$: a centroaffine surface

K: the Euclidean Gaussian curvature < 0

(x, y): asymptotic line coordinates

$\psi := h(\partial_x, \partial_y)$

d: the signed distance from the origin to the tangent plane

$$\rho := -\frac{1}{4} \log \left(-\frac{K}{d^4} \right)$$

$$\alpha := \psi \det \begin{pmatrix} f \\ f_x \\ f_{xx} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_x \\ f_y \end{pmatrix}$$

$$\beta := \psi \det \begin{pmatrix} f \\ f_y \\ f_{yy} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_y \\ f_x \end{pmatrix}$$

Gauss formula in asymptotic line coordinates

Gauss formula

$$\begin{cases} f_{xx} = \left(\dfrac{\psi_x}{\psi} + \rho_x\right) f_x + \dfrac{\alpha}{\psi} f_y \\ f_{xy} = -\psi f + \rho_y f_x + \rho_x f_y \\ f_{yy} = \left(\dfrac{\psi_y}{\psi} + \rho_y\right) f_y + \dfrac{\beta}{\psi} f_x \end{cases}$$

Proof

Use

$$\Gamma^x_{xy} = -\frac{1}{4}\frac{K_y}{K}, \quad \Gamma^y_{xy} = -\frac{1}{4}\frac{K_x}{K}$$

etc.

Integrability conditions

Proposition

The integrability conditions for the above Gauss formula are

$$\begin{cases} (\log\psi)_{xy} = -\psi - \dfrac{\alpha\beta}{\psi^2} + \rho_x\rho_y \\ \alpha_y + \rho_x\psi_x = \rho_{xx}\psi \\ \beta_x + \rho_y\psi_y = \rho_{yy}\psi \end{cases}$$

If ρ is constant and $\alpha, \beta \neq 0$, changing the coordinates, if necessary, we obtain Tzitzéica equation:

$$(\log\psi)_{xy} = -\psi - \frac{1}{\psi^2}$$

Proposition

T: the Tchebychev vector field

$\Longrightarrow T = \mathrm{grad}_h\rho$

Equations of associativity

$f : M \to \mathbf{R}^3$: an indefinite centroaffine surface
(x, y): asymptotic line coordinates
κ: the curvature of the centroaffine metric h

$$\kappa = -\frac{(\log \psi)_{xy}}{\psi}$$

Consider the flat case. ($\kappa \equiv 0$)
May assume $\psi \equiv 1$.
$\Longrightarrow$ Equation of associativity in topological field theory:

$$g_{xxx}g_{yyy} - g_{xxy}g_{xyy} + 1 = 0$$

$$(\rho = g_{xy},\ \alpha = g_{xxx},\ \beta = g_{yyy})$$

2004 E. V. Ferapontov: Showed that the integrability conditions for flat centroaffine hypersurfaces are equivalent to equations of associativity.

Centroaffine surfaces with vanishing Tchebychev operator

$f = (X, Y, Z) : M \to \mathbf{R}^3$: a centroaffine surface
$\nabla^h T$: the Tchebychev operator
κ: the curvature of the centroaffine metric h
$\nabla^h T \equiv 0 \Longrightarrow \kappa \equiv 0$ except proper affine spheres centered at the origin with $\kappa \equiv 1$

The case $\kappa \equiv 0$: $p, q, r \in \mathbf{R}$

- $X^p Y^q Z^r = 1$
- $\left\{ \exp \left(-p \tan^{-1} \dfrac{X}{Y} \right) \right\} (X^2 + Y^2)^q Z^r = 1$
- $Z = -X(p \log X + q \log Y)$
- $Z = \pm X \log X + \dfrac{Y^2}{X}$
- $f = (e^x, A_1(x)e^y, A_2(x)e^y)$, A_1, A_2: Linearly independent solutions to the differential equation: $A'' - A' - a(x)A = 0$, a: an arbitrary function

Another example

Theorem (F. 2014)

$f : M \to \mathbf{R}^3$: a flat centroaffine surface

$\nabla^h T$: the Tchebychev operator

$J := \frac{1}{2}\|C\|^2 = \frac{1}{2}h^{ip}h^{jq}h_{kr}C^k_{ij}C^r_{pq}$: the Pick function

$\nabla^h T$: nonsemisimple & J: constant

$$\Longrightarrow f = \left(\sum_{n=0}^{\infty}\varphi_{n,1}(x)y^n, \sum_{n=0}^{\infty}\varphi_{n,2}(x)y^n, \sum_{n=0}^{\infty}\varphi_{n,3}(x)y^n\right)$$

(x, y): asymptotic line coordinates around $(x_0, 0)$ s.t. $x_0 \neq 0$

$\varphi_{0,1}, \varphi_{0,2}, \varphi_{0,3}$: Linearly independent solutions to the differential equation: $x\varphi''' + \varphi'' - \varphi = 0$

$(n+1)\varphi_{n+1,i} = x\varphi''_{n,i}$ $(i = 1, 2, 3)$

Characterizations of the surface

- $J \equiv -1$
- Solutions to the equation of associativity:
  $$g = \frac{1}{6}xy^3 - \frac{1}{2}x^2 \log x + (\text{polynomials with degree} \leq 2)$$
- Centroaffine minimal: Extremals of the area integral of the centroaffine metric h
  $\iff \operatorname{tr} \nabla^h T \equiv 0$
- The differential equation can be solved by using Meijer G functions:
  $$\varphi = c_1 G^{2,0}_{0,3}\left(\frac{x^2}{8}\,\middle|\,\begin{matrix}-\\ \frac{1}{2}, \frac{1}{2}, 0\end{matrix}\right) + ic_2 G^{1,0}_{0,3}\left(-\frac{x^2}{8}\,\middle|\,\begin{matrix}-\\ \frac{1}{2}, \frac{1}{2}, 0\end{matrix}\right)$$
  $$+ c_3 G^{1,0}_{0,3}\left(-\frac{x^2}{8}\,\middle|\,\begin{matrix}-\\ 0, \frac{1}{2}, \frac{1}{2}\end{matrix}\right) \quad (c_1, c_2, c_3 \in \mathbf{R})$$
  The second and the third terms can be written by using the generalized hypergeometric function ${}_0F_2$.

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