

中心アファイン曲面に対する3次形式の 平行性と再帰性

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- Cubic form: A fundamental invariant for affine hypersurfaces
- Equiaffine differential geometry
 - Maschke-Pick-Berwald's Theorem:
A Blaschke hypersurface with vanishing cubic form is a piece of a quadric.
 - 1989 Nomizu-Pinkall:
If the cubic form of a Blaschke surface does not vanish and is parallel relative to the induced connection, the surface is a piece of a Cayley surface: $z = xy - \frac{1}{3}x^3$.

 - A graph of a cubic polynomial
 - A ruled surface
 - Equiaffinely homogeneous
 - An improper affine sphere

Problem

Problem

Characterize fundamental centroaffine surfaces by the cubic form.

- Examples of fundamental centroaffine surfaces
 - Quadrics
 - A ruled surface given by

$$f(x, y) = A'(x) + yA(x)$$

A is an \mathbf{R}^3 -valued function s.t. $\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0$.

- The curvature of the centroaffine metric is 1.
- The Pick function vanishes.
- Centroaffine minimal
- Projective minimal

Euclidean differential geometry and affine differential geometry

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Consider hypersurfaces in the Euclidean space.

- Euclidean differential geometry
 - \exists a unit normal vector field
 - Study properties invariant under the Euclidean transformation.
- Affine differential geometry
 - Choose a transversal vector field.
 - Equiaffine differential geometry
 - Take a Blaschke normal vector field.
 - Study properties invariant under the equiaffine transformation.
 - Centroaffine differential geometry
 - Take a radial vector field.
 - Study properties invariant under the affine transformation fixing the origin.

Gauss-Weingarten formula

Euclidean differential geometry

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$f : M^n \rightarrow \mathbf{R}^{n+1}$: a hypersurface

D : the standard flat connection on \mathbf{R}^{n+1}

$X, Y \in \mathfrak{X}(M)$

○ Euclidean differential geometry

n : a unit normal vector field

\implies Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y)n & \text{(Gauss)} \\ D_X n = -f_* SX & \text{(Weingarten)} \end{cases}$$

∇ : the Levi-Civita connection

h : the second fundamental form

S : the shape operator

Gauss-Weingarten formula

Affine differential geometry

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$f : M^n \rightarrow \mathbf{R}^{n+1}$: a hypersurface

D : the standard flat connection on \mathbf{R}^{n+1}

$X, Y \in \mathfrak{X}(M)$

○ Affine differential geometry

ξ : a transversal vector field

\implies Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y) \xi & \text{(Gauss)} \\ D_X \xi = -f_* S X + \tau(X) \xi & \text{(Weingarten)} \end{cases}$$

f is called an affine hypersurface.

∇ : the induced connection

h : the affine fundamental form

Considered as a metric.

S : the affine shape operator

τ : the transversal connection form

Nondegenerate, definite or indefinite affine hypersurfaces

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Definition

$f : M^n \rightarrow \mathbf{R}^{n+1}$: an affine hypersurface

h : the affine fundamental form

f : nondegenerate (resp. definite, indefinite)

\Updownarrow def.

h : nondegenerate (resp. definite, indefinite)

Proposition

The above definition is independent of the choice of the transversal vector field ξ .

Proof

$$\bar{\xi} := \varphi\xi + f_*Z \quad (\varphi : M \rightarrow \mathbf{R} \setminus \{0\}, Z \in \mathfrak{X}(M)) \implies \varphi\bar{h} = h$$

Cubic form

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∇ : the induced connection

h : the affine fundamental form

τ : the transversal connection form

$X, Y, Z \in \mathfrak{X}(M)$

Codazzi equation

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z)$$

The cubic form:

$$C(X, Y, Z) := (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z)$$

Defines a symmetric $(0, 3)$ -tensor.

Definition of Blaschke hypersurfaces

$f : M^n \rightarrow \mathbf{R}^{n+1}$: a nondegenerate affine hypersurface

θ : the volume form induced by the immersion f

$$\theta(X_1, \dots, X_n) := \omega(f_*X_1, \dots, f_*X_n, \xi) \quad (X_1, \dots, X_n \in \mathfrak{X}(M))$$

ω : a parallel volume form with respect to the flat connection D

ω_h : the volume form with respect to the affine fundamental form h

$$X_1, \dots, X_n \in \mathfrak{X}(M) \text{ s.t. } \theta(X_1, \dots, X_n) = 1$$

$$\omega_h(X_1, \dots, X_n) := |\det(h(X_i, X_j))|^{\frac{1}{2}}$$

Proposition

$\exists \xi$ (unique up to the sign) s.t. $\tau = 0$ & $\theta = \omega_h$

The above ξ is called a Blaschke normal vector field.

f is called a Blaschke hypersurface.

h is called a Blaschke metric.

Characterizations by the cubic form

$f : M^n \rightarrow \mathbf{R}^{n+1}$: a Blaschke hypersurface

C : the cubic form

Maschke-Pick-Berwald's theorem

$C = 0 \implies f$: a piece of a quadric

∇ : the induced connection

Theorem (Nomizu-Pinkall 1989)

$n = 2, \nabla C = 0, C \neq 0 \implies f$: a piece of a Cayley surface

1988 Vrancken: $n = 3, \nabla C = 0, C \neq 0$

$\hat{\nabla}$: the Levi-Civita connection for the affine fundamental form h

1989 Magid-Nomizu: $n = 2, \hat{\nabla} C = 0, C \neq 0$

- Example: a Cayley surface

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Definition

$f : M^n \rightarrow \mathbf{R}^{n+1}$: a hypersurface

f : a centroaffine hypersurface

\Updownarrow def.

The radial vector intersects with the tangent space transversally at any point.

$$\xi := - \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$$

Weingarten formula:

$$D_X \xi = -f_* X$$

\Updownarrow

$$S = \text{the identity}, \tau = 0$$

The affine fundamental form h is called a centroaffine metric.

Fundamental facts

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Consider nondegenerate centroaffine hypersurfaces.

$f : M^n \rightarrow \mathbf{R}^{n+1}$: a centroaffine hypersurface

C : the cubic form

Proposition

$C = 0 \implies f$: a piece of a quadric centered at the origin

Remark

In general, a nondegenerate affine hypersurface with $\tau = 0$ & $C = 0$ is a piece of a quadric.

∇ : the induced connection

$\hat{\nabla}$: the Levi-Civita connection for the centroaffine metric h

1991 Li-Wang: $\hat{\nabla}C = 0$ & flat

2015 Hildebrand: $\hat{\nabla}C = 0$ ($\nabla C = 0$)

Tchebychev form

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$f : M^n \rightarrow \mathbf{R}^{n+1}$: a centroaffine hypersurface

h : the centroaffine metric

C : the cubic form

$X \in \mathfrak{X}(M)$

Define a $(1, 1)$ -tensor A_X by

$$C(X, Y, Z) = h(A_X(Y), Z) \quad (\forall Y, Z \in \mathfrak{X}(M)).$$

Define a 1-form $\text{tr}_h C$ by

$$(\text{tr}_h C)(X) = \text{tr} A_X.$$

$T := \frac{1}{n} \text{tr}_h C$ is called a Tchebychev form.

Traceless part of the cubic form

$f : M^n \rightarrow \mathbf{R}^{n+1}$: a centroaffine hypersurface

h : the centroaffine metric

C : the cubic form

T : the Tchebychev form

$X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned}\tilde{C}(X, Y, Z) := & C(X, Y, Z) - \frac{n}{n+2}(T(X)h(Y, Z) \\ & + T(Y)h(Z, X) + T(Z)h(X, Y)) \\ & \text{(traceless part of } C\text{)}\end{aligned}$$

Remark

\tilde{C} coincides with the cubic form as a Blaschke hypersurface.

$\hat{\nabla}$: the Levi-Civita connection for h

1997 Liu-Wang: $n = 2$, $\hat{\nabla} \tilde{C} = 0$

Definiteness and the Euclidean Gaussian curvature

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Proposition

$f : M^2 \rightarrow \mathbf{R}^3$: a centroaffine surface

f : definite (resp. indefinite)



the Euclidean Gaussian curvature: positive (resp. negative)

Proof

(x_1, x_2) : local coordinates

Gauss formula:

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} - h(\partial_{x_i}, \partial_{x_j})f \quad (i = 1, 2)$$

Take the inner product with the unit normal vector field.

Notations

Consider the indefinite case.

$f : M^2 \rightarrow \mathbf{R}^3$: a centroaffine surface

K : the Euclidean Gaussian curvature < 0

(u, v) : asymptotic line coordinates

$\varphi := h(\partial_u, \partial_v)$

d : the signed distance from the origin to the tangent plane

$$\rho := -\frac{1}{4} \log \left(-\frac{K}{d^4} \right)$$

$$a := \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} / \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}$$

$$b := \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} / \det \begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix}$$

Gauss formula in asymptotic line coordinates

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Gauss formula

$$\begin{cases} f_{uu} = \left(\frac{\varphi_u}{\varphi} + \rho_u \right) f_u + \frac{a}{\varphi} f_v \\ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v \\ f_{vv} = \left(\frac{\varphi_v}{\varphi} + \rho_v \right) f_v + \frac{b}{\varphi} f_u \end{cases}$$

Proposition

The integrability conditions for the above Gauss formula are

$$\begin{cases} (\log \varphi)_{uv} = -\varphi - \frac{ab}{\varphi^2} + \rho_u \rho_v \\ a_v + \rho_u \varphi_u = \rho_{uu} \varphi \\ b_u + \rho_v \varphi_v = \rho_{vv} \varphi \end{cases}$$

Covariant derivative of the cubic form

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Lemma

$f : M^2 \rightarrow \mathbf{R}^3$: an indefinite centroaffine surface

∇ : the induced connection

C : the cubic form

$\nabla C = 0 \iff$ All the functions below vanish.

$$-a_u + \frac{3a\varphi_u}{\varphi} + 6a\rho_u, \quad -a_v + 3a\rho_v + 3\rho_u^2\varphi,$$

$$-b_v + \frac{3b\varphi_v}{\varphi} + 6b\rho_v, \quad -b_u + 3b\rho_u + 3\rho_v^2\varphi,$$

$$3a\rho_v + \rho_u\varphi_u - \rho_{uu}\varphi + 3\rho_u^2\varphi, \quad 3b\rho_u + \rho_v\varphi_v - \rho_{vv}\varphi + 3\rho_v^2\varphi,$$

$$\frac{ab}{\varphi} + 5\rho_u\rho_v\varphi - \rho_{uv}\varphi$$

Main results

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$f : M^2 \rightarrow \mathbf{R}^3$: a centroaffine surface

∇ : the induced connection

C : the cubic form

Theorem 1 (F.-Hamamoto-Nakai)

$\nabla C = 0 \implies f$: a piece of a quadric centered at the origin
(an ellipsoid or a hyperboloid)

Proof

If f is indefinite, show that $C = 0$ by use of the integrability conditions and the above Lemma.

Then f is a piece of a hyperboloid of one sheet centered at the origin.

If f is definite, similar argument as above shows that f is a piece of an ellipsoid or a hyperboloid of two sheets.

Main results (continued)

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∇ : the induced connection

\tilde{C} : the traceless part of the cubic form

Theorem 2 (F.-Hamamoto-Nakai)

\tilde{C} : recurrent relative to ∇ :

$$\tilde{C} \neq 0 \ \& \ \exists \sigma \in \Omega^1(M) \text{ s.t. } \nabla \tilde{C} = \sigma \otimes \tilde{C}$$

$\implies f$: a piece of a ruled surface given by

$$f(x, y) = A'(x) + yA(x)$$

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