Centroaffine surfaces with parallel or recurrent cubic form relative to the induced connection

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Contents

Backgrounds

- Cubic form: A fundamental invariant for affine hypersurfaces
- Equiaffine differential geometry
  - Maschke-Pick-Berwald's Theorem:
    A Blaschke hypersurface with vanishing cubic form is a piece of a quadric.
  - 1989 Nomizu-Pinkall:
    If the cubic form of a Blaschke surface does not vanish and is parallel relative to the induced connection, the surface is a piece of a Cayley surface: $z = xy - \frac{1}{3}x^3$.

    · A graph of a cubic polynomial
    · A ruled surface
    · Equiaffinely homogeneous
    · An improper affine sphere

Problem

Problem

Characterize fundamental centroaffine surfaces by the cubic form.

- Examples of fundamental centroaffine surfaces
  - Quadrics
  - A ruled surface given by

$$f(x, y) = A'(x) + yA(x)$$

A is an $\mathbf{R}^3$-valued function s.t. $\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0.$

  - · The curvature of the centroaffine metric is 1.
  - · The Pick function vanishes.
  - · Centroaffine minimal
  - · Projective minimal

Euclidean differential geometry and affine differential geometry

Consider hypersurfaces in the Euclidean space.

- Euclidean differential geometry
  - $^{\exists}$a unit normal vector field
  - Study properties invariant under the Euclidean transformation.
- Affine differential geometry
  - Choose a transversal vector field.
    - Equiaffine differential geometry
      - Take a Blaschke normal vector field.
      - Study properties invariant under the equiaffine transformation.
    - Centroaffine differential geometry
      - Take a radial vector field.
      - Study properties invariant under the affine transformation fixing the origin.

Gauss-Weingarten formula

Euclidean differential geometry

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

D: the standard flat connection on $\mathbf{R}^{n+1}$

$X, Y \in \mathfrak{X}(M)$

- Euclidean differential geometry

  n: a unit normal vector field

  $\Longrightarrow$ Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y) n & \text{(Gauss)} \\ D_X n = -f_* SX & \text{(Weingarten)} \end{cases}$$

  ∇: the Levi-Civita connection

  h: the second fundamental form

  S: the shape operator

Gauss-Weingarten formula

Affine differential geometry

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

D: the standard flat connection on $\mathbf{R}^{n+1}$

$X, Y \in \mathfrak{X}(M)$

- Affine differential geometry

  ξ: a transversal vector field

  $\Longrightarrow$ Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi & \text{(Gauss)} \\ D_X \xi = -f_* SX + \tau(X)\xi & \text{(Weingarten)} \end{cases}$$

  f is called an affine hypersurface.

  ∇: the induced connection

  h: the affine fundamental form

  Considered as a metric.

  S: the affine shape operator

  τ: the transversal connection form

Nondegenerate, definite or indefinite affine hypersurfaces

Definition

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface
h: the affine fundamental form

$$f\text{: nondegenerate (resp. definite, indefinite)}$$
$$\Updownarrow \text{ def.}$$
$$h\text{: nondegenerate (resp. definite, indefinite)}$$

Proposition

The above definition is independent of the choice of the transversal vector field ξ.

Proof

$$\bar{\xi} := \varphi\xi + f_* Z \ (\varphi : M \to \mathbf{R} \setminus \{0\},\ Z \in \mathfrak{X}(M)) \implies \varphi\bar{h} = h$$

Cubic form

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface
∇: the induced connection
h: the affine fundamental form
τ: the transversal connection form
$X, Y, Z \in \mathfrak{X}(M)$

Codazzi equation

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z)$$

The cubic form:

$$C(X, Y, Z) := (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z)$$

Defines a symmetric $(0, 3)$-tensor.

Definition of Blaschke hypersurfaces

$f : M^n \to \mathbf{R}^{n+1}$: a nondegenerate affine hypersurface

θ: the volume form induced by the immersion f

$$\theta(X_1, \ldots, X_n) := \omega(f_* X_1, \ldots, f_* X_n, \xi) \quad (X_1, \ldots, X_n \in \mathfrak{X}(M))$$

ω: a parallel volume form with respect to the flat connection D

ω_h: the volume form with respect to the affine fundamental form h

$$X_1, \ldots, X_n \in \mathfrak{X}(M) \text{ s.t. } \theta(X_1, \ldots, X_n) = 1$$

$$\omega_h(X_1, \ldots, X_n) := |\det(h(X_i, X_j))|^{\frac{1}{2}}$$

Proposition

$^\exists \xi$ (unique up to the sign) s.t. $\tau = 0$ & $\theta = \omega_h$

The above ξ is called a Blaschke normal vector field.

f is called a Blaschke hypersurface.

h is called a Blaschke metric.

Characterizations by the cubic form

$f : M^n \to \mathbf{R}^{n+1}$: a Blaschke hypersurface
C: the cubic form

Maschke-Pick-Berwald's theorem

$C = 0 \Longrightarrow f$: a piece of a quadric

∇: the induced connection

Theorem (Nomizu-Pinkall 1989)

$n = 2,\ \nabla C = 0,\ C \neq 0 \Longrightarrow f$: a piece of a Cayley surface

1988 Vrancken: $n = 3,\ \nabla C = 0,\ C \neq 0$

$\hat{\nabla}$: the Levi-Civita connection for the affine fundamental form h
1989 Magid-Nomizu: $n = 2,\ \hat{\nabla} C = 0,\ C \neq 0$
2011 Hu-Li-Vrancken: n: arbitrary & h: definte
2011 Hu-Li-Li-Vrancken: n: arbitrary & h: Lorentzian

Definition of centroaffine hypersurfaces

Definition

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

f: a centroaffine hypersurface

$\Updownarrow$ def.

The radial vector intersects with the tangent space transversally at any point.

$$\xi := -\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$$

Weingarten formula:

$$D_X \xi = -f_* X$$

$$\Updownarrow$$

$$S = \text{the identity}, \ \tau = 0$$

The affine fundamental form h is called a centroaffine metric.

Fundamental facts

Consider nondegenerate centroaffine hypersurfaces.
$f : M^n \to \mathbf{R}^{n+1}$: a centroaffine hypersurface
C: the cubic form

Proposition

$C = 0 \implies f$: a piece of a quadric centered at the origin

∇: the induced connection
$\hat{\nabla}$: the Levi-Civita connection for the centroaffine metric h
1991 Li-Wang: $\hat{\nabla} C = 0$ & flat
2015 Hildebrand: $\hat{\nabla} C = 0$ $(\nabla C = 0)$
2017 Cheng-Hu-Moruz: $\hat{\nabla} C = 0$ & h: definite (geometric)

Tchebychev form

$f : M^n \to \mathbf{R}^{n+1}$: a centroaffine hypersurface
h: the centroaffine metric
C: the cubic form
$X \in \mathfrak{X}(M)$
Define a $(1,1)$-tensor A_X by

$$C(X, Y, Z) = h(A_X(Y), Z) \quad ({}^\forall Y, Z \in \mathfrak{X}(M)).$$

Define a 1-form $\mathrm{tr}_h C$ by

$$(\mathrm{tr}_h C)(X) = \mathrm{tr} A_X.$$

$T := \frac{1}{n}\mathrm{tr}_h C$ is called a Tchebychev form.

Traceless part of the cubic form

$f : M^n \to \mathbf{R}^{n+1}$: a centroaffine hypersurface
h: the centroaffine metric
C: the cubic form
T: the Tchebychev form
$X, Y, Z \in \mathfrak{X}(M)$

$$\tilde{C}(X, Y, Z) := C(X, Y, Z) - \frac{n}{n+2}(T(X)h(Y, Z) + T(Y)h(Z, X) + T(Z)h(X, Y))$$

(traceless part of C)

Remark

$\tilde{C}$ coincides with the cubic form as a Blaschke hypersurface.

$\hat{\nabla}$: the Levi-Civita connection for h
1997 Liu-Wang: $n = 2$, $\hat{\nabla}\tilde{C} = 0$

Definiteness and the Euclidean Gaussian curvature

Proposition

$f : M^2 \to \mathbf{R}^3$: a centroaffine surface

$$f\text{: definite (resp. indefinite)}$$
$$\Updownarrow$$
$$\text{the Euclidean Gaussian curvature: positive (resp. negative)}$$

Proof

(x_1, x_2): local coordinates

Gauss formula:

$$f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} - h(\partial_{x_i}, \partial_{x_j}) f \quad (i = 1, 2)$$

Take the inner product with the unit normal vector field.

Notations

Consider the indefinite case.

$f : M^2 \to \mathbf{R}^3$: a centroaffine surface

K: the Euclidean Gaussian curvature < 0

(u, v): asymptotic line coordinates

$\varphi := h(\partial_u, \partial_v)$

d: the signed distance from the origin to the tangent plane

$$\rho := -\frac{1}{4} \log \left(-\frac{K}{d^4} \right)$$

$$a := \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}$$

$$b := \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix}$$

Gauss formula in asymptotic line coordinates

Gauss formula

$$\begin{cases} f_{uu} = \left(\dfrac{\varphi_u}{\varphi} + \rho_u\right) f_u + \dfrac{a}{\varphi} f_v \\ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v \\ f_{vv} = \left(\dfrac{\varphi_v}{\varphi} + \rho_v\right) f_v + \dfrac{b}{\varphi} f_u \end{cases}$$

Proposition

The integrability conditions for the above Gauss formula are

$$\begin{cases} (\log \varphi)_{uv} = -\varphi - \dfrac{ab}{\varphi^2} + \rho_u \rho_v \\ a_v + \rho_u \varphi_u = \rho_{uu} \varphi \\ b_u + \rho_v \varphi_v = \rho_{vv} \varphi \end{cases}$$

Covariant derivative of the cubic form

Lemma

$f : M^2 \to \mathbf{R}^3$: an indefinite centroaffine surface
∇: the induced connection
C: the cubic form

$$\nabla C = 0 \iff \text{All the functions below vanish.}$$

$$-a_u + \frac{3a\varphi_u}{\varphi} + 6a\rho_u, \ -a_v + 3a\rho_v + 3\rho_u^2\varphi,$$

$$-b_v + \frac{3b\varphi_v}{\varphi} + 6b\rho_v, \ -b_u + 3b\rho_u + 3\rho_v^2\varphi,$$

$$3a\rho_v + \rho_u\varphi_u - \rho_{uu}\varphi + 3\rho_u^2\varphi, \ 3b\rho_u + \rho_v\varphi_v - \rho_{vv}\varphi + 3\rho_v^2\varphi,$$

$$\frac{ab}{\varphi} + 5\rho_u\rho_v\varphi - \rho_{uv}\varphi$$

Main results

$f : M^2 \to \mathbf{R}^3$: a centroaffine surface
∇: the induced connection
C: the cubic form

Theorem 1 (F.-Hamamoto-Nakai)

$\nabla C = 0 \Longrightarrow f$: a piece of a quadric centered at the origin (an ellipsoid or a hyperboloid)

Proof

If f is indefinite, show that $C = 0$ by use of the integrability conditions and the above Lemma.
Then f is a piece of a hyperboloid of one sheet centered at the origin.
If f is definite, similar argument as above shows that f is a piece of an ellipsoid or a hyperboloid of two sheets.

Main results (continued)

$f : M^2 \to \mathbf{R}^3$: a centroaffine surface
∇: the induced connection
$\tilde{C}$: the traceless part of the cubic form

Theorem 2 (F.-Hamamoto-Nakai)

$\tilde{C}$: recurrent relative to ∇:

$$\tilde{C} \neq 0 \ \& \ {}^{\exists}\sigma \in \Omega^1(M) \text{ s.t. } \nabla \tilde{C} = \sigma \otimes \tilde{C}$$

$\Longrightarrow$ f: a piece of a ruled surface given by

$$f(x, y) = A'(x) + yA(x)$$

Thank you for your attention.