Centroaffine surfaces of cohomogeneity one

Atsushi Fujioka

Faculty of Engineering Science
Kansai University

September 3, 2019
Ritsumeikan University
The 5th Japan-China Geometry Conference
Joint work with H. Furuhata

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Backgrounds

Problem

Consider an analogue of surfaces of revolution in affine differential geometry.

Hypersurfaces in the affine space: various choices of a transversal vector field

- Blaschke normal vector field $\rightsquigarrow$ Blaschke hypersurfaces
- Restriction of the radial vector field $\rightsquigarrow$ centroaffine hypersurfaces

Equiaffine rotation surfaces:

an analogue of surfaces of revolution for Blaschke surfaces
Blaschke normal vector fields meet a fixed line or are parallel to a fixed plane.

Centroaffine surfaces: Consider surfaces of cohomogeneity one.

Centroaffine differential geometry and centroaffine surfaces of cohomogeneity one

Centroaffine differential geometry:

Study properties of submanifolds in the affine space which are invariant by centroaffine transformations.

Centroaffine transformation: affine transformation fixing the origin

$$\rightsquigarrow$$

Group of centroaffine transformations of $\mathbf{R}^n \cong \mathrm{GL}(n, \mathbf{R})$

A centroaffine surface of cohomogeneity one is expressed as

$$f(x, y) = \gamma(x) e^{yA},$$

where

$$\gamma\text{: a curve in } \mathbf{R}^3, \quad A \in \mathfrak{gl}(3, \mathbf{R}).$$

Example 1

Flat proper affine spheres centered at the origin

Example 1 (cf. 1990 Magid-Ryan)

$f = (X, Y, Z) : M^2 \to \mathbf{R}^3$: a flat proper affine sphere centered at the origin

$\Longrightarrow$ Up to centroaffine congruence,

$$XYZ = 1 \quad \text{or} \quad (X^2 + Y^2)Z = 1.$$

These are of cohomogeneity one.
For example, in the former case,

$$\gamma(x) = (e^x, 1, e^{-x}), \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The above surfaces are included in the next example.

Example 2

Nondegenerate flat centroaffine surfaces with vanishing Tchebychev operator

Example 2 (1995 Liu-Wang)

$f = (X, Y, Z) : M^2 \to \mathbf{R}^3$: a nondegenerate flat centroaffine surface, Tchebychev operator $= 0$

$\Longrightarrow$ Up to centroaffine congruence, f is one of the following:

(1) $X^p Y^q Z^r = 1$ $(p, q, r \in \mathbf{R},\ pqr(p + q + r) \neq 0)$,

(2) $\left\{\exp\left(-p \tan^{-1} \frac{X}{Y}\right)\right\}(X^2 + Y^2)^q Z^r = 1$ $(p, q, r \in \mathbf{R},\ r(2q + r)(p^2 + q^2) \neq 0)$,

(3) $Z = -X(p \log X + q \log Y)$ $(p, q \in \mathbf{R},\ q(p + q) \neq 0)$,

(4) $Z = \pm X \log X + \frac{Y^2}{X}$,

(5) $f = (e^x, \psi_1(x)e^y, \psi_2(x)e^y)$, ψ_1, ψ_2 are linearly independent solutions to the differential equation $\psi'' - \psi' - \lambda\psi = 0$ for an arbitrary function $\lambda = \lambda(x)$.

These are of cohomogeneity one.

The Gauss-Weingarten formula

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface (immersion)
ξ: a vector field along f

$$(f, \xi) \text{ or } f\text{: an affine hypersurface}$$
$$\Updownarrow \text{ def.}$$
$$^\forall x \in M,\ \xi(x) \pitchfork f_* T_x M$$

ξ is called a transversal vector field.

Let f be an affine hypersurface.
D: the standard flat connection on $\mathbf{R}^{n+1}$
$\Longrightarrow$ The Gauss-Weingarten formula:

$$\begin{cases} D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi \\ D_X \xi = -f_* SX + \tau(X)\xi \end{cases} \qquad (X, Y \in \mathfrak{X}(M))$$

∇: the induced connection, h: the affine fundamental form
S: the affine shape operator, τ: the transversal connection form

Integrability conditions

The fundamental equations or the Gauss-Codazzi-Ricci equations

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface

$X, Y, Z \in \mathfrak{X}(M) \Longrightarrow$

The Gauss equation:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY$$

(R: the curvature tensor for the induced connection ∇)

The Codazzi equation for h:

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z)$$

The Codazzi equation for S:

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX$$

The Ricci equation:

$$h(X, SY) - h(SX, Y) = d\tau(X, Y)$$

These are the integrability conditions for the Gauss-Weingarten formula.

Equiaffine hypersurfaces

Relative normalization

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface
ξ: a transversal vector field
Fix a volume element ω on $\mathbf{R}^{n+1}$ which is parallel w.r.t. the standard flat connection D.
$\rightsquigarrow$ θ: the volume element induced by (f, ξ)

$$\theta(X_1, \ldots, X_n) = \omega(f_* X_1, \ldots, f_* X_n, \xi) \ (X_1, \ldots, X_n \in \mathfrak{X}(M))$$

Proposition

$$\nabla_X \theta = \tau(X)\theta \quad (X \in \mathfrak{X}(M))$$

Definition

$$\xi \text{ or } f\text{: equiaffine} \underset{\text{def.}}{\iff} \nabla\theta = 0 \iff \tau = 0$$

When the Euclidean metric is given, the unit normal vector field is equiaffine.

Fundamental facts about equiaffine hypersurfaces 1/2

Proposition

$f : M^n \to \mathbf{R}^{n+1}$: an equiaffine hypersurface
$n \geq 2$
$S = \lambda\,\mathrm{id.}$ $({}^\exists\lambda : M \to \mathbf{R})$
$\Longrightarrow \lambda$: constant

Proof

By equiaffineness, the Codazzi equation for S becomes

$$(\nabla_X S)(Y) = (\nabla_Y S)(X).$$

Substitute $S = \lambda\,\mathrm{id.}$ into the above equation, we have

$$(X\lambda)Y = (Y\lambda)X.$$

In the following, assume $n \geq 2$.

Fundamental facts about equiaffine hypersurfaces 2/2

Proposition

$f : M^n \to \mathbf{R}^{n+1}$: an equiaffine hypersurface

(1) $S = 0 \iff$ Transversal vector fields are parallel to each other.

(2) $S = \lambda\,\mathrm{id.}$ ($\lambda \in \mathbf{R} \setminus \{0\}$) $\iff$ Transversal vector fields meet a fixed point (the center).

Proof of $\Rightarrow$ in (2)

By equiaffineness, the Weingarten formula becomes

$$D_X \xi = -f_* SX.$$

$$g(x) := f(x) + \frac{1}{\lambda}\xi(x) \quad (x \in M)$$

$\Longrightarrow D_X g = 0$

Nondegenerate, definite, indefinite affine hypersurfaces

Definition

$f : M^n \to \mathbf{R}^{n+1}$: an affine hypersurface

h: the affine fundamental form

$$f\text{: nondegenerate (resp. definite, indefinite)}$$

$$\Updownarrow \text{def.}$$

$$h\text{: nondegenerate (resp. definite, indefinite)}$$

Proposition

The above definition is independent of a choice of a transversal vector filed ξ.

Proof

$\bar{\xi} := \lambda\xi + (\text{tangential part}) \quad (\lambda : M \to \mathbf{R} \setminus \{0\}) \implies \lambda\bar{h} = h$

Definition of Blaschke hypersurfaces

$f : M^n \to \mathbf{R}^{n+1}$: a nondegenerate affine hypersurface
ξ: a transversal vector field
τ: the transversal connection form
θ: the volume form induced by (f, ξ)

The affine fundamental form h is nondegenerate.
$\rightsquigarrow \omega_h$: the volume form w.r.t. h

Proposition

There exists a transversal vector field ξ with $\tau = 0$, $\theta = \omega_h$, which is unique up to sign (the Blaschke normal vector field).

In particular, the Blaschke normal vector field is equiaffine.
In this case, f is called a Blaschke hypersurface.
h is called the Blaschke metric.

Affine hyperspheres

Definition

$f : M^n \to \mathbf{R}^{n+1}$: a Blaschke hypersurface

S: the affine shape operator

$$f\text{: an affine hypersphere} \underset{\text{def.}}{\iff} S\text{: a scalar operator}$$

Improper affine hyperspheres: affine hyperspheres with $S = 0$
Blaschke normal vector fields are parallel to each other.

Proper affine hyperspheres: affine hyperspheres with $S \neq 0$
Blaschke normal vector fields meet a fixed point (the center).

Examples of affine hyperspheres

Example (proper quadrics)

- ∘ Proper quadrics without center are improper affine hyperspheres.
- ∘ Proper quadrics with center are proper affine hyperspheres.

Theorem (1990 Magid-Ryan, cf. Example 1)

$f = (X, Y, Z) : M^2 \to \mathbf{R}^3$: an affine sphere, the curvature of the Blaschke metric $= 0$ (flat)

$\Longrightarrow$ Up to affine congruence, f is one of the following:

- ∘ $Z = X^2 + Y^2$ (definite, improper),
- ∘ $Z = XY + \lambda(X)$, λ: an arbitrary function of X (indefinite, improper),
- ∘ $XYZ = 1$ (definite, proper),
- ∘ $(X^2 + Y^2)Z = 1$ (indefinite, proper).

Definition of centroaffine hypersurfaces and the difference tensor

Definition

$f : M^n \to \mathbf{R}^{n+1}$: a hypersurface

$$\xi := -\left(\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}\right)\Bigg|_f$$

f: a centroaffine hypersurface

$\Updownarrow$ def.

(f, ξ): an affine hypersurface

$f : M^n \to \mathbf{R}^{n+1}$: a nondegenerate centroaffine hypersurface
∇: the induced connection
h: the affine fundamental form (the centroaffine metric)
$\hat{\nabla}$: the Levi-Civita connection for h
$C := \nabla - \hat{\nabla}$: the difference tensor

The Tchebychev vector field and the Tchebychev operator

$T := \frac{1}{n}\mathrm{tr}_h C$: the Tchebychev vector field

Proposition

$$T = 0$$

$$\Updownarrow$$

f: As a Blaschke hypersurface, the affine shape operator is a nonzero scalar operator.
Blaschke normal vector fields meet the origin.
(a proper affine hypersphere centered at the origin)

$\mathcal{T} := \hat{\nabla} T$: the Tchebychev operator

1995 Liu-Wang: Classified nondegenerate centroaffine surfaces with $T \neq 0$, $\mathcal{T} = 0$ (cf. Example 2).

Centroaffine minimal hypersurfaces

$f_t : M^n \to \mathbf{R}^{n+1}$: a one parameter family of nondegenerate centroaffine hypersurfaces s.t.

$$f_0 = f \text{ and } f_t = f, \ \operatorname{Im}(f_t)_* = \operatorname{Im} f_* \text{ on the boundary of } M$$

$$\left.\frac{\partial f_t}{\partial t}\right|_{t=0} = \lambda f + (\text{tangential part}) \quad (\lambda : M \to \mathbf{R})$$

ω_{h_t}: the volume form w.r.t. the centroaffine metric h_t of f_t

The first variation formula (1994 Wang)

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \omega_{h_t} = \pm\frac{n}{2}\int_M \lambda\,(\operatorname{tr}\mathcal{T})\,\omega_h$$

Definition (1994 Wang)

$$f\text{: centroaffine minimal} \underset{\text{def.}}{\iff} \operatorname{tr}\mathcal{T} = 0$$

Definite, indefinite affine surfaces and the Euclidean Gaussian curvature

Proposition

$f : M^2 \to \mathbf{R}^3$: a nondegenerate affine surface
K: the Euclidean Gaussian curvature

$$f\text{: definite (resp. indefinite)} \iff K > 0 \text{ (resp. } K < 0)$$

Proof

(x_1, x_2): a local coordinate
The Gauss formula becomes

$$f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + h(\partial_{x_i}, \partial_{x_j})\xi \quad (i = 1, 2).$$

n: the unit normal vector field
Taking the inner product of the above equation and n, we have

$$\langle f_{x_i x_j}, n\rangle = h(\partial_{x_i}, \partial_{x_j})\langle \xi, n\rangle.$$

The normal form for nondegenerate centroaffine surfaces of cohomogeneity one

For simplicity, consider the indefinite case.

Lemma

$f: M^2 \to \mathbf{R}^3$: an indefinite centroaffine surface cohomogeneity one

$\Longrightarrow$ f can be expressed as

$$f(u,v) = \gamma(u+v)e^{(u-v)A} = \gamma(s)e^{tA}$$

(the normal form), where

(u,v): an asymptotic line coordinate,

γ: a curve in $\mathbf{R}^3$, $\quad A \in \mathfrak{gl}(3,\mathbf{R})$,

$$s = u+v, \quad t = u-v.$$

Notations

$f : M^2 \to \mathbf{R}^3$: an indefinite centroaffine surface

(u, v): an asymptotic line coordinate

$\varphi := h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$

d: the Euclidean support function from the origin

$$\rho := -\frac{1}{4} \log\left(-\frac{K}{d^4}\right)$$

$(= \log(\pm r))$

(r: the equiaffine support function from the origin)

$$a := \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}$$

$$b := \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix}$$

The Gauss formula for the normal form

$f : M^2 \to \mathbf{R}^3$: an indefinite centroaffine surface
cohomogeneity one
expressed in the normal form

$\Longrightarrow$ The Gauss formula:

$$\begin{cases} f_{uu} = \left(\dfrac{\varphi'}{\varphi} + \alpha' + \dfrac{1}{2}C_1\right) f_u + \dfrac{a}{\varphi} f_v, \\ f_{vv} = \left(\dfrac{\varphi'}{\varphi} + \alpha' - \dfrac{1}{2}C_1\right) f_v + \dfrac{b}{\varphi} f_u, \\ f_{uv} = -\varphi f + \left(\alpha' - \dfrac{1}{2}C_1\right) f_u + \left(\alpha' + \dfrac{1}{2}C_1\right) f_v, \end{cases}$$

where

φ, α, a, b: functions of s, $\quad \rho(s,t) = \alpha(s) + \dfrac{C_1}{2} t, \quad C_1 = \operatorname{tr} A.$

The integrability conditions for the normal form

The integrability conditions for the normal form:

$$\begin{cases} \left(\dfrac{\varphi'}{\varphi}\right)' = -\varphi - \dfrac{ab}{\varphi^2} + (\alpha')^2 - \dfrac{1}{4}C_1^2 \\ a' + \left(\alpha' + \dfrac{1}{2}C_1\right)\varphi' = \alpha''\varphi \\ b' + \left(\alpha' - \dfrac{1}{2}C_1\right)\varphi' = \alpha''\varphi \end{cases}$$

From the second and the third equations, ${}^\exists\lambda = \lambda(s)$, ${}^\exists C_2 \in \mathbf{R}$ s.t.

$$\lambda' = \alpha''\varphi - \alpha'\varphi', \quad a = -\frac{1}{2}C_1\varphi + \lambda, \quad b = \frac{1}{2}C_1\varphi + \lambda - C_2.$$

By a further computation, we can construct nondegenerate centroaffine surfaces of cohomogeneity one.

Special cases

Proposition

$f : M^2 \to \mathbf{R}^3$: an indefinite centroaffine surface
cohomogeneity one
expressed in the normal form

(1) $C_1 = C_2 = 0$
$\Longrightarrow$ Up to centroaffine congruence, f is one of the following equiaffine rotation surfaces:

$$f(x, y) = \begin{cases} (\mu(x) \cos y, \mu(x) \sin y, x), \\ (\mu(x) \cosh y, \mu(x) \sinh y, x), \\ \left(x, xy, \frac{1}{2}xy^2 + \mu(x)\right). \end{cases}$$

(2) f: a proper affine sphere centered at the origin
$\Longleftrightarrow$ α: constant, $C_1 = 0$

(3) f: centroaffine minimal $\Longleftrightarrow$ α: a linear function

The case of nonflat proper affine spheres

Theorem (F.-Furuhata)

$f : M^2 \to \mathbf{R}^3$: an indefinite nonflat proper affine sphere
centered at the origin
cohomogeneity one

$\Longrightarrow$ In the normal form, γ and A are given by the following:

- ∘ The minimal polynomial of A is $t^3 - \dfrac{c}{4}t + \dfrac{a-b}{8}$, where $a, b, c \in \mathbf{R}$.
- ∘ γ is a solution to the differential equation

$$(2\varphi' + a + b)\gamma' = \gamma\{4\varphi A^2 + (a-b)A - 2\varphi^2 E\},$$

where $\varphi \neq 0$, $\varphi' \neq 0, -\frac{a+b}{2}$ and

$$(\varphi')^2 = -2\varphi^3 + c\varphi^2 + ab.$$

The case of centroaffine miminal surfaces with constant curvature

Theorem (F.-Furuhata)

$f : M^2 \to \mathbf{R}^3$: a centroaffine minimal surface
cohomogeneity one

$\kappa :=$ the curvature of the centroaffine metric: constant

$\Longrightarrow \kappa = 0, 1$

Devided into theree cases.

Expressed explicitly.

- $\circ$ $\kappa = 0$, $\mathcal{T} = 0$ (cf. 1995 Liu-Wang)
- $\circ$ $\kappa = 1$ and ruled (cf. 2009 F: without assumption of cohomogeneity one) or a part of an ellipsoid (definite)
- $\circ$ Other surfaces with $\kappa = 1$ (cf. 2006 F: special cases)

Thank you for your attention.