Equivariant projections between spaces of equicentroaffine curves

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Backgrounds and main results

- The space of closed equicentroaffine plane curves
  - $^\exists$Action of the diffeomorphism group of the circle
  - Considered as an action of the Virasoro-Bott group.
  - The space is considered as the coadjoint orbit of the dual of the Virasoro algebra.
  - Studied from the viewpoint of symplectic geometry.
- Today: Consider the space of equicentroaffine curves in general vector space.
  - $^\exists$Action of the diffeomorphism group of the line
  - Define projections into the space of plane or space curves.
  - The above projections are equivariant w.r.t. the above action.

Definition of the equicentroaffine curve

Definition

I: an interval

$n = 2, 3, 4, \ldots$

$$\gamma : I \to \mathbf{R}^n \setminus \{0\}\text{: an equicentroaffine curve}$$

$$\Updownarrow \text{def.}$$

$$\det \begin{pmatrix} \gamma \\ \gamma' \\ \vdots \\ \gamma^{(n-1)} \end{pmatrix} = 1$$

$s \in I$ is called an equicentroaffine arclength parameter.

The fundamental theorem of equicentroaffine curves

$\gamma : I \to \mathbf{R}^n \setminus \{0\}$: an equicentroaffine curve
$\Longrightarrow {}^{\exists}\kappa_1, \kappa_2, \ldots, \kappa_{n-1} : I \to \mathbf{R}$ s.t.

$$\gamma^{(n)} + \kappa_1\gamma^{(n-2)} + \kappa_2\gamma^{(n-3)} + \cdots + \kappa_{n-1}\gamma = 0$$

For $i = 1, 2, \ldots, n-1$, we call κ_i the i-th curvature.

The fundamental theorem of equicentroaffine curves

$\kappa_1, \kappa_2, \ldots, \kappa_{n-1} : I \to \mathbf{R}$
$\Longrightarrow {}^{\exists}\gamma : I \to \mathbf{R}^n \setminus \{0\}$: an equicentroaffine curve with the i-th curvature κ_i
Unique up to equiaffine transformation fixing the origin.

Equiaffine transformation fixing the origin:

- Multiplication by the element of $SL(n, \mathbf{R})$
- Called equicentroaffine transformation.

Example

Equicentroaffine plane curves with constant curvature

Consider equicentroaffine plane curves.
We call the first curvature the equicentroaffine curvature.

Example 1 (Equicentroaffine plane curves with constant curvature)

$\gamma : I \to \mathbf{R}^2 \setminus \{0\}$: an equicentroaffine plane curve
κ: the equicentroaffine curvature

○ $\kappa = 0 \iff \gamma$ is a part of a line:

$$\gamma(s) = (a + bs, c + ds) \quad (a,\ b,\ c,\ d \in \mathbf{R},\ ad - bc = 1)$$

○ κ: a positive constant $\iff \gamma$ is a part of an ellipse:

$$\gamma(s) = \left(a\cos\frac{s}{ab}, b\sin\frac{s}{ab}\right) \quad \left(a,\ b > 0,\ \kappa = \frac{1}{a^2b^2}\right)$$

○ κ: a negative constant $\iff \gamma$ is a part of a hyperbola.

Definition of the action of the diffeomorphism group of the line

- $\mathcal{M}_n$: the set of all equicentroaffine curves from $\mathbf{R}$ into $\mathbf{R}^n \setminus \{0\}$
- $\mathcal{M}_n/\mathrm{SL}(n, \mathbf{R})$: the set of all congruence classes of equicentroaffine curves from $\mathbf{R}$ into $\mathbf{R}^n \setminus \{0\}$
- $\mathrm{Diff}\,(\mathbf{R})$: the group of all orientation preserving diffeomorphisms of $\mathbf{R}$
- $\gamma \in \mathcal{M}_n$, $g \in \mathrm{Diff}\,(\mathbf{R})$

$$\tilde{\gamma}(s) := (\gamma \cdot g)(s) := (g'(s))^{\frac{1-n}{2}}(\gamma \circ g)(s) \quad (s \in \mathbf{R})$$

Proposition

$\gamma \cdot g$ defines an action of $\mathrm{Diff}\,(\mathbf{R})$ on $\mathcal{M}_n$, $\mathcal{M}_n/\mathrm{SL}(n, \mathbf{R})$.

Proof

For $k = 0, 1, 2, \ldots, n-1$,

$$\tilde{\gamma}^{(k)} = (g')^{\frac{1-n}{2}+k}(\gamma^{(k)} \circ g) + \cdots .$$

Notations

- Consider the action of $\mathrm{Diff}\,(\mathbf{R})$ on $\mathcal{M}_n$:

$$\mathcal{M}_n \ni \gamma \mapsto \tilde{\gamma} = \gamma \cdot g \in \mathcal{M}_n \quad (g \in \mathrm{Diff}\,(\mathbf{R}))$$

- Derive the transformation rule for the curvatures.
- $\alpha := \dfrac{1-n}{2}$
- Write $\gamma^{(k)} \circ g$ simply as $\gamma^{(k)}$.
- $h := \dfrac{g''}{g'}$
- Consider h, h', h'', $\dots$ as independent variables.
- Define the degree of $h^{(k)}$ as $(k+1)$.
- Define the weighted degree of a polynomial P of h, h', h'', $\dots$.
- Denote the weighted degree by $\deg_w P$.

Transformation rule for the derivatives of curves 1/3

Lemma

- For $k = 1, 2, \ldots, n$,

$$\tilde{\gamma}^{(k)} = \sum_{l=0}^{k-1} P_{k,l}\tilde{\gamma}^{(l)} + (g')^{\alpha+k}\gamma^{(k)},$$

  where $P_{k,l}$'s are homogeneous polynomials of $h, h', h'', \ldots$ s.t.

$$\deg_w P_{k,l} = k - l.$$

- The following three recurrence relations hold:
  - For $k = 1, 2, \ldots, n-1$,

$$P_{k+1,0} = \sum_{m=0}^{k-1} \frac{\partial P_{k,0}}{\partial h^{(m)}} h^{(m+1)} - (\alpha + k)P_{k,0}h. \qquad \text{(R1)}$$

Transformation rule for the derivatives of curves 2/3

Lemma (continued)

- For $k = 2, 3, \ldots, n-1$, $l = 1, 2, \ldots, k-1$,

$$P_{k+1,l} = \sum_{m=0}^{k-l-1} \frac{\partial P_{k,l}}{\partial h^{(m)}} h^{(m+1)} + P_{k,l-1} - (\alpha + k) P_{k,l} h. \quad \text{(R2)}$$

- For $k = 1, 2, \ldots, n-1$,

$$P_{k+1,k} = P_{k,k-1} + (\alpha + k) h. \quad \text{(R3)}$$

Proof

First, differentiating the equation

$$\tilde{\gamma} = (g')^{\alpha} \gamma,$$

we have

$$\tilde{\gamma}' = \alpha h \tilde{\gamma} + (g')^{\alpha+1} \gamma'.$$

Transformation rule for the derivatives of curves 3/3

Proof (continued)

Hence we have

$$P_{1,0} = \alpha h.$$

$\tilde{\gamma}'$ is expressed as above.
$P_{1,0}$ is a homogeneous polynomial s.t. $\deg_w P_{1,0} = 1$.

Next, for $k = 1, 2, \ldots, n-1$, assume that

- $\tilde{\gamma}^{(k)}$'s are expressed as above.
- $P_{k,l}$'s are homogeneous polynomials s.t. $\deg_w P_{k,l} = k - l$.

∘ Compute $\tilde{\gamma}^{(k+1)}$ using that $P_{k,l}$ is a polynomial of h, h', $\ldots$, $h^{(k-l-1)}$.
∘ Recurrence relations are derived.
∘ Compute $\deg_w P_{k+1,l}$.

Equivariant projections between spaces of equicentroaffine curves

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Example 2 ($P_{k+1,k}$)

$k = 1, 2, \ldots, n-1$

From $P_{1,0} = \alpha h$ and (R3):

$$P_{k+1,k} = P_{k,k-1} + (\alpha + k)h,$$

we have

$$P_{k+1,k} = P_{1,0} + \sum_{l=1}^{k} (\alpha + l)h$$
$$= (k+1)\left(\alpha + \frac{k}{2}\right) h.$$

Since $\alpha = \frac{1-n}{2}$, we have

$$P_{n,n-1} = 0. \quad \text{(Assume the weighted degree is 1.)}$$

Example

$P_{2,0}$, $P_{3,0}$

Example 3 ($P_{2,0}$, $P_{3,0}$)

From $P_{1,0} = \alpha h$ and (R1):

$$P_{k+1,0} = \sum_{m=0}^{k-1} \frac{\partial P_{k,0}}{\partial h^{(m)}} h^{(m+1)} - (\alpha + k) P_{k,0} h,$$

we have

$$\begin{aligned} P_{2,0} &= \frac{\partial P_{1,0}}{\partial h} h' - (\alpha + 1) P_{1,0} h \\ &= \alpha h' - \alpha(\alpha + 1) h^2. \end{aligned}$$

Moreover, if $n \geq 3$, we have

$$P_{3,0} = \alpha h'' - \alpha(3\alpha + 4) h h' + \alpha(\alpha + 1)(\alpha + 2) h^3.$$

Transformation rule for the curvatures

1/3

Theorem 1

$\gamma \in \mathcal{M}_n$

κ_i: the i-th curvature of γ $(i = 1, 2, \ldots, n-1)$

$\tilde{\gamma} = \gamma \cdot g$ $(g \in \mathrm{Diff}(\mathbf{R}))$

$\tilde{\kappa}_i$: the i-th curvature of $\tilde{\gamma}$

$\Longrightarrow$ For $l = 0, 1, 2, \ldots, n-3$,

$$\tilde{\kappa}_{n-l-1} = (g')^{n-l}\kappa_{n-l-1} - P_{n,l} - \sum_{k=l+1}^{n-2} (g')^{n-k}\kappa_{n-k-1}P_{k,l}.$$

$$\tilde{\kappa}_1 = (g')^2\kappa_1 + \frac{n(n^2-1)}{12}S(g),$$

where $S(g)$ is the Schwarzian derivative of g:

$$S(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2.$$

Proof

First, by use of the transformation rule for the derivatives of curves (Lemma), we have

$$\tilde{\gamma}^{(n)} = \sum_{l=0}^{n-2} P_{n,l}\tilde{\gamma}^{(l)} - (g')^n \kappa_{n-1}\tilde{\gamma} - \sum_{k=1}^{n-2}(g')^{n-k}\kappa_{n-k-1}\left(\tilde{\gamma}^{(k)} - \sum_{l=0}^{k-1} P_{k,l}\tilde{\gamma}^{(l)}\right).$$

Hence we have the first equation in Theorem 1 and

$$\tilde{\kappa}_1 = (g')^2\kappa_1 - P_{n,n-2}.$$

Next, compute $P_{n,n-2}$.

Transformation rule for the curvatures 3/3

Proof (continued)

From (R2) and Example 2 ($P_{k+1,k}$), we have

$$\begin{aligned} P_{k+1,k-1} - P_{k,k-2} &= \frac{\partial P_{k,k-1}}{\partial h} h' - (\alpha + k) P_{k,k-1} h \\ &= k\left(\alpha + \frac{k-1}{2}\right) h' \\ &\quad - (\alpha + k) k \left(\alpha + \frac{k-1}{2}\right) h^2. \end{aligned}$$

Moreover, from Example 3 ($P_{2,0}$), we have

$$P_{n,n-2} = -\frac{n(n^2-1)}{12} S(g).$$

Equivariant projections into $\mathcal{M}_2$

1/2

$n = 3, 4, 5, \dots$

$\gamma \in \mathcal{M}_n$

κ_1: the first curvature of γ

$\bar{\gamma} \in \mathcal{M}_2$: an equicentroaffine plane curve s.t.

$$\text{the equicentroaffine curvature} = \frac{6}{n(n^2-1)}\kappa_1$$

Consider the action of $\mathrm{Diff}(\mathbf{R})$.

Theorem 2

The correspondence from γ to $\bar{\gamma}$ defines an equivariant map from $\mathcal{M}_n$ into $\mathcal{M}_2$:

$$\overline{\gamma \cdot g} = \bar{\gamma} \cdot g \quad (g \in \mathrm{Diff}(\mathbf{R})).$$

Equivariant projections into $\mathcal{M}_2$

1/2

Proof

By Theorem 1, the transformation rule for the first curvature is given by

$$\tilde{\kappa}_1 = (g')^2 \kappa_1 + \frac{n(n^2-1)}{12} S(g).$$

$\bar{\kappa}_1$, $\bar{\tilde{\kappa}}_1$: the equicentroaffine curvature of $\bar{\gamma}$, $\overline{\gamma \cdot g}$

Then

$$\frac{n(n^2-1)}{6} \bar{\tilde{\kappa}}_1 = (g')^2 \cdot \frac{n(n^2-1)}{6} \bar{\kappa}_1 + \frac{n(n^2-1)}{12} S(g)$$

$$\Updownarrow$$

$$\bar{\tilde{\kappa}}_1 = (g')^2 \bar{\kappa}_1 + \frac{1}{2} S(g)$$

This is the transformation rule for the equicentroaffine curvature when $n = 2$.

Example 4 (Equicentroaffine curves with vanishing higher curvatures)

$n = 3, 4, 5, \ldots$

$\rho : \mathbf{R} \to \mathbf{R}^2 \setminus \{0\}$: a curve s.t. $\rho^{(n-2)}$ is an equicentroaffine plane curve with the equicentroaffine curvature κ

Define an equicentroaffine curve $\gamma : \mathbf{R} \to \mathbf{R}^n \setminus \{0\}$ by

$$\gamma(s) = \left(1, s, \frac{1}{2!}s^2, \ldots, \frac{1}{(n-3)!}s^{n-3}, \rho\right) \quad (s \in \mathbf{R}).$$

$\Longrightarrow \kappa_1 = \kappa,\ \kappa_2 = \kappa_3 = \cdots = \kappa_{n-1} = 0$

$\Longrightarrow$ The projection into $\mathcal{M}_2$ is given by

$$\bar{\gamma}(s) = \rho^{(n-2)}\left(\sqrt{\frac{6}{n(n^2-1)}}s\right) \quad (s \in \mathbf{R}).$$

Equivariant projections into $\mathcal{M}_3$ 1/3

$n = 4, 5, 6, \ldots$
$\gamma \in \mathcal{M}_n$
κ_1, κ_2: the first and the second curvature of γ
$\bar{\gamma} \in \mathcal{M}_3$: an equicentroaffine space curve s.t.

$$\text{the first curvature} = \frac{24}{n(n^2-1)}\kappa_1$$

$$\text{the second curvature} = \frac{24}{n(n^2-1)(n-2)}\kappa_2$$

Consider the action of $\mathrm{Diff}(\mathbf{R})$.

Theorem 3

The correspondence from γ to $\bar{\gamma}$ defines an equivariant map from $\mathcal{M}_n$ to $\mathcal{M}_3$.

Equivariant projections into $\mathcal{M}_3$
2/3

Proof

By Theorem 1, the transformation rule for the first curvature is given by

$$\tilde{\kappa}_1 = (g')^2\kappa_1 + \frac{n(n^2-1)}{12}S(g).$$

The transformation rule for the second curvature is given by

$$\tilde{\kappa}_2 = (g')^3\kappa_2 - P_{n,n-3} - (g')^2\kappa_1 P_{n-2,n-3}.$$

From Example 2 ($P_{k+1,k}$), we have

$$P_{n-2,n-3} = -(n-2)h.$$

Equivariant projections into $\mathcal{M}_3$ 2/3

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Proof (continued)

From (R2), we have

$$P_{k+1,k-2} = \frac{\partial P_{k,k-2}}{\partial h} h' + \frac{\partial P_{k,k-2}}{\partial h'} h'' + P_{k,k-3} - (\alpha + k) P_{k,k-2} h.$$

Hence we have

$$\begin{aligned} P_{n,n-3} &= P_{3,0} \\ &\quad + \sum_{k=3}^{n-1} \left(\frac{\partial P_{k,k-2}}{\partial h} h' + \frac{\partial P_{k,k-2}}{\partial h'} h'' - (\alpha + k) P_{k,k-2} h \right). \end{aligned}$$

Further computation shows that

$$P_{n,n-3} = -\frac{n(n^2-1)(n-2)}{24} (S(g))'.$$

Example

Closed equicentroaffine curves with constant curvatures when n is even

Example 5 (Closed equicentroaffine curves with constant curvatures when n is even)

$\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{N}$, $\lambda_i \neq \lambda_j$ $(i \neq j)$
Define an equicentroaffine curve $\gamma : \mathbf{R} \to \mathbf{R}^{2m} \setminus \{0\}$ by

$$\gamma(s) = (\cos \lambda_1 s, \sin \lambda_1 s, \ldots, \cos \lambda_m s, \mu \sin \lambda_m s) \quad (s \in \mathbf{R}),$$

where

$$\frac{1}{\mu} = \prod_{i=1}^{m} \lambda_i \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2.$$

Then

$$t^{2m} + \kappa_1 t^{2m-2} + \cdots + \kappa_{2m-1} = (t^2 + \lambda_1^2) \cdots (t^2 + \lambda_m^2)$$

$$(\kappa_2 = \kappa_4 = \cdots = \kappa_{2m-2} = 0).$$

Example

Closed equicentroaffine curves with constant curvatures when n is even (continued)

Example 5 (continued)

$l \in \mathbf{N}$

$$\lambda_1 := l, \quad \lambda_2 := 3l, \quad \dots, \quad \lambda_m := (2m-1)l$$

$$\Downarrow$$

$$\frac{6}{2m\{(2m)^2-1\}}\kappa_1 = l^2$$

Hence the projection into $\mathcal{M}_2$ is given by

$$\bar{\gamma}(s) = \left(\cos ls, \frac{1}{l}\sin ls\right) \quad (s \in \mathbf{R}).$$

Moreover, the projection into $\mathcal{M}_3$ is given by

$$\bar{\gamma}(s) = \left(\cos 2ls, \sin 2ls, \frac{1}{8l^3}\right) \quad (s \in \mathbf{R}).$$

Example

Closed equicentroaffine curves with constant curvatures when n is odd

Example 6 (Closed equicentroaffine curves with constant curvatures when n is odd)

$\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{N}$, $\lambda_i \neq \lambda_j$ $(i \neq j)$
Define an equicentroaffine curve $\gamma : \mathbf{R} \to \mathbf{R}^{2m+1} \setminus \{0\}$ by

$$\gamma(s) = (\cos \lambda_1 s, \sin \lambda_1 s, \ldots, \cos \lambda_m s, \sin \lambda_m s, \mu) \quad (s \in \mathbf{R}),$$

where

$$\frac{1}{\mu} = \prod_{i=1}^{m} \lambda_i^3 \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2.$$

Then

$$t^{2m+1} + \kappa_1 t^{2m-1} + \cdots + \kappa_{2m} = t(t^2 + \lambda_1^2) \cdots (t^2 + \lambda_m^2)$$

$$(\kappa_2 = \kappa_4 = \cdots = \kappa_{2m} = 0).$$

Example

Closed equicentroaffine curves with constant curvatures when n is odd (continued)

Example 6 (continued)

$l \in \mathbf{N}$

$$\lambda_1 := 2l, \quad \lambda_2 := 4l, \quad \ldots, \quad \lambda_m := 2ml$$

$$\Downarrow$$

$$\frac{6}{(2m+1)\{(2m+1)^2-1\}}\kappa_1 = l^2$$

Hence the projection into $\mathcal{M}_2$ is given by

$$\bar{\gamma}(s) = \left(\cos ls, \frac{1}{l}\sin ls\right) \quad (s \in \mathbf{R}).$$

Moreover, the projection into $\mathcal{M}_3$ is given by

$$\bar{\gamma}(s) = \left(\cos ls, \sin ls, \frac{1}{l^3}\right) \quad (s \in \mathbf{R}).$$

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Thank you for your attention.