A maximal domain for strategy-proof and no-vetoer rules in the multi-object choice model

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Abstract. Following Barberà, Sonnenschein, and Zhou (1991, Econometrica 59, 595-609), we study rules (or social choice functions) through which agents select a subset from a set of objects. We investigate domains on which there exist nontrivial strategy-proof rules. We establish that the set of separable preferences is a maximal domain for the existence of rules satisfying strategy-proofness and no-vetoer.

Keywords. social choice, mechanism design, voting by committees, generalized median voter scheme, separable preference

JEL Classification Codes. C72, D71, H41

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1 Introduction

Situations exist in which agents choose a subset from a set of objects. For example, existing members of a club choose new members from a list of candidates, and city council members choose public projects to carry out from a list. Barberà et al. (1991) model these situations and axiomatically examine a rule (or a social choice function) that maps each preference profile to a subset of objects. They first assume that agents' preferences satisfy **separability**, which requires that an object e is preferred to the null outcome if and only if any set of objects including e is preferred to that set subtracting e. We refer to the class of separable preferences as the **separable domain**. Barberà et al. (1991) establish that on the separable domain, a class of rules called "voting by committees" satisfies "strategy-proofness" and "ontoness", and only this class satisfies those requirements. **Strategy-proofness**, which is one of the most frequently employed properties for incentive compatibility, requires that no agent can be better off by misrepresenting her true preference, whatever preferences other agents may have. **Ontoness**, which is recognized as a minimal requirement for agent sovereignty, requires that any subset of objects can be an outcome for some preference profile. Thus, their result is positive in the sense that the class of voting by committees includes a variety of rules, all of which satisfy both requirements. Their model and result are followed in various studies.[1]

The larger the domain of rules, the greater the variety of situations to which the results can be applied. Thus, once we obtain a positive result on some domain, we wish to enlarge the domain as long as the positive result holds. However, in this model, Gibbard (1973) and Satterthwaite's (1975) theorem implies that if the domain is unrestricted, no rule other than trivial ones such as dictatorships satisfies *strategy-proofness* and *ontoness*. A natural question then arises: *(i) how large can the domain be while the class of voting by committees satisfies strategy-proofness and ontoness*? Because the class of voting by committees includes trivial rules such as dictatorships, which satisfy both requirements on the unrestricted domain, this question is qualified as: *(i*) how large can the domain be while nontrivial voting by committees satisfies strategy-proofness and ontoness*? Barberà et al. (1991) themselves address this problem, and establish that the separable domain is a maximal domain where voting by "no-vetoer" committees satisfies strategy-proofness.[2] **No-vetoer** is a condition that excludes trivial rules such as dictatorships. It says that no agent has a veto power, and this is sufficient for *ontoness*.[3]

Note that in the search for maximal domains, we need not restrict rules to a specific class of rules such as voting by committees, a priori, because there may be other interesting rules. Restricting rules to voting by committees in the search for maximal domains might make the

[1]See, for example, Shimomura (1996), Ju (2003, 2005), Berga et al. (2004, 2006), Barberà et al. (2005), and Nehring and Puppe (2007).

[2]In fact, the rule employed by Barberà et al. (1991) is voting by no-vetoer and "no-dummy" committees. *No-dummy* is employed to make all agents' sets of preferences equal. In this paper, we omit this condition because we assume exogenously that all agents' sets of preferences are the same. Similar types of the maximal domain problem for various rules are studied by Serizawa (1995), Barberà et al. (1999), and Berga (2002) for the generalized median voter scheme; Barbie et al. (2006) and Vorsatz (2008) for Borda's rule; and Sanver (2009) for the plurality rule.

[3]*No-vetoer* is employed in various studies including Repullo (1987), Maskin (1999), and Berga and Serizawa (2000).

maximal domains unnecessarily small. This is why we search for maximal domains without restricting the rules to voting by committees. We generalize the above question *(i*)* as: *(ii) how large can the domain be while there exists a nontrivial rule satisfying strategy-proofness and ontoness*? Berga and Serizawa (2000) study this general maximal domain problem *(ii)* in the model where the set of alternatives is a continuous line. Many authors study this type of maximal domain problem in various models, including Ching and Serizawa (1998), Massó and Neme (2001, 2004), Ehlers (2002), and Mizobuchi and Serizawa (2006). However, no author has investigated the general maximal domain problem in the original multi-object choice model by Barberà et al. (1991). In this paper, we establish that *the separable domain is a maximal domain for the existence of rules satisfying strategy-proofness and no-vetoer*. Although we seek a larger domain than the separable domain by not assuming voting by committees, this result states that they coincide. As we discuss the details in the Appendix, the general maximal domain problem in the multi-object choice model requires us to develop a much more complex proof procedure than in the previous literature.

The rest of this paper is organized as follows. Section 2 sets out the details of the model. Section 3 states the main theorem. Section 4 notes some remaining questions and concludes. The Appendix includes the proof for the main theorem.

2 Preliminaries

Let $N \equiv \{1, \cdots, n\}$ be the set of **agents** (or voters). Assume $n \geq 3$.[4] A **coalition** is a subset I of N, and let $\#I$ denote the number of agents in I. Let $K \equiv \{1, \cdots, k\}$ be the set of **objects**. Let Z denote the set of **alternatives** that are the vertices of a k-dimensional hypercube; that is, $Z \equiv \prod_{e=1}^{k} Z_e$, where for all $e \in K$, $Z_e \equiv \{0, 1\}$. Given $z \in Z$ and $e \in K$, $z_e = 0$ represents that the object e is not selected and $z_e = 1$ represents that the object e is selected.[5] We endow Z with the L_1-norm. That is, for every $y, z \in Z$,

$$||y - z|| \equiv \sum_{e=1}^{k} |y_e - z_e|.$$

Given $y, z \in Z$, the **box** containing y and z is defined as

$$B(y, z) \equiv \{x \in Z : ||y - z|| = ||y - x|| + ||x - z||\}.$$

Preferences are complete, transitive, and asymmetric binary relations over Z. Generic preferences without links to a specific agent are denoted by P_0, P_0', $\hat{P}_0$, and so on. Agent i's preferences are denoted by P_i, P_i', $\hat{P}_i$, and so on. Let $\mathcal{D}_U$ denote the set of all preferences. We call the n-tuple of sets of all preferences $\mathcal{D}_U^n$ the **universal domain**. Given $P_0 \in \mathcal{D}_U$, let $\tau(P_0) \in Z$ be such that for all $z \in Z \backslash \{\tau(P_0)\}$, $\tau(P_0)\, P_0\, z$. We call $\tau(P_0)$ the **top** for P_0. A

[4]In the following investigation, we impose "no-vetoer" on rules. This property is not meaningful if there are only two agents.

[5]Our representation of an alternative follows Barberà et al. (1993), which studies a more general model than ours.

preference profile is defined as $P \equiv (P_1, \ldots, P_n) \in \mathcal{D}_U^n$. For $i, j \in N$, let $(P_i', P_{-i}) \in \mathcal{D}_U^n$ denote the preference profile obtained from P by replacing P_i with P_i', $(P_j'', P_i', P_{-\{i,j\}}) \in \mathcal{D}_U^n$ denote the profile obtained from (P_i', P_{-i}) by replacing P_j with P_j'', and so on. Given a coalition $I \subseteq N$, let $P_I \in \mathcal{D}_U^{\#I}$ denote a $\#I$-tuple of preferences associated with I, and $P_{-I} \in \mathcal{D}_U^{n-\#I}$ denote an $(n - \#I)$-tuple of preferences associated with $N \backslash I$. Let $\tau(P) \equiv (\tau(P_1), \ldots, \tau(P_n))$, which is the profile of tops associated with P. A **domain** is a subset $\mathcal{D}^n$ of $\mathcal{D}_U^n$. A **rule** (or a social choice function) on a domain $\mathcal{D}^n$ is defined as a function $f : \mathcal{D}^n \to Z$. Note that we implicitly deal with the case where the domains of all agents' preferences can be considered as the same.

"Separability" of preferences is usually defined as that for all $e \in K$ and all $X \subseteq K \backslash \{e\}$, $X \cup \{e\} \, P_0 \, X \iff \{e\} \, P_0 \, \varnothing$. By using the notions of alternative and box, this also can be represented as follows.

Separability. For all $y, z \in Z$ such that $y \neq z$ and $y \in B(z, \tau(P_0))$, $y \, P_0 \, z$.

In the following investigation, we employ the latter representation of separability.[6] Let $\mathcal{D}_S$ denote the set of separable preferences. We call $\mathcal{D}_S^n$ the **separable domain**.

We introduce several basic properties of a rule. The first prevents agents from gaining by misrepresenting their true preferences. The second says that any alternative can be an outcome. The third forbids the rule from giving any agent an extreme decisive power. The fourth forbids the rule from giving any agent an extreme veto power.

Strategy-proofness. For all $P \in \mathcal{D}^n$, all $i \in N$, and all $\hat{P}_i \in \mathcal{D} \backslash \{P_i\}$, $f(P) \, P_i \, f(\hat{P}_i, P_{-i})$ or $f(P) = f(\hat{P}_i, P_{-i})$.

Ontoness. For all $z \in Z$, there exists $P \in \mathcal{D}^n$ such that $f(P) = z$.

No-dictator. There is no $i \in N$ such that for all $P \in \mathcal{D}^n$, $f(P) = \tau(P_i)$.

No-vetoer. There is no $z \in Z$, $i \in N$, and $P_i \in \mathcal{D}$ such that for all $P_{-i} \in \mathcal{D}^{n-1}$, $f(P) \neq z$.

If f fails *strategy-proofness*, f is said to be **manipulable**. Furthermore, we say that agent i manipulates f at P via $\hat{P}_i$ if $f(\hat{P}_i, P_{-i}) \, P_i \, f(P)$. *No-vetoer* is equivalent to that for all $i \in N$, all $z \in Z$, and all $P_i \in \mathcal{D}$, there exists $P_{-i} \in \mathcal{D}^{n-1}$ such that $f(P) = z$. Also note that *no-vetoer* implies both *ontoness* and *no-dictator*.

Next, we introduce a class of rules, which Barberà et al. (1991) call "voting by committees", that plays an important role in our paper. A coalition is said to be "winning" for an object e if it has the power to have the object e selected. Voting by committees is a rule generated by specifying the class of winning coalitions for each object. We assume that for each object, (1) the empty coalition is not winning, (2) the set of all agents is winning, and (3) larger coalitions have more power.

Set of winning coalitions $\mathcal{W}_e \subsetneq 2^N$ for an object $e \in K$. (1) $\varnothing \notin \mathcal{W}_e$, (2) $N \in \mathcal{W}_e$, and (3) for all $I, I' \in 2^N$ such that $I \in \mathcal{W}_e$ and $I \subseteq I'$, $I' \in \mathcal{W}_e$.

[6] The way to restrict preferences using box first appears in Barberà et al. (1993), which studies a more general model than ours. In the same multi-object choice model, Barberà et al. (2005) employ the representation of separability using box.

Given $\mathcal{W}_e$, let $\underline{\mathcal{W}}_e \equiv \{I \in \mathcal{W}_e : \text{for all } i \in I, I \backslash \{i\} \notin \mathcal{W}_e\}$, which we call the set of **minimal winning coalitions** associated with $\mathcal{W}_e$. A **winning coalition system** is defined as $\mathcal{W} \equiv \{\mathcal{W}_e\}_{e=1}^k$. Voting by committees is a rule associated with a winning coalition system such that each object e is selected in the outcome if and only if the set of agents whose top alternative contains e belongs to the set of winning coalitions for e.

Voting by committees. There exists a winning coalition system $\mathcal{W}$ such that for all $P \in \mathcal{D}^n$ and all $e \in K$,

$$f_e(P) = 1 \iff \{i \in N : \tau_e(P_i) = 1\} \in \mathcal{W}_e.$$

The following is the main result by Barberà et al. (1991).

Theorem 1 (Barberà et al., 1991)**.** *A rule on the separable domain satisfies strategy-proofness and ontoness if and only if it is voting by committees.*[7]

Immediately, we obtain the characterization result by using *strategy-proofness* and *no-vetoer* on the separable domain as a corollary of Theorem 1, which must be a strict subset of the set of voting by committees. The characterized rules are defined by a winning coalition system and additionally satisfy (1) any sole agent cannot be a winning coalition, and (2) any coalition with $n-1$ members is a winning coalition.

Set of no-vetoer winning coalitions $\mathcal{W}_e \subsetneq 2^N$ for an object $e \in K$. (1) For all $i \in N, \{i\} \notin \mathcal{W}_e$, (2) for all $i \in N, N \backslash \{i\} \in \mathcal{W}_e$, and (3) for all $I, I' \in 2^N$ such that $I \in \mathcal{W}_e$ and $I \subseteq I'$, $I' \in \mathcal{W}_e$.

Voting by no-vetoer committees. There exists a no-vetoer winning coalition system $\mathcal{W}$ such that for all $P \in \mathcal{D}^n$ and all $e \in K$,

$$f_e(P) = 1 \iff \{i \in N : \tau_e(P_i) = 1\} \in \mathcal{W}_e$$

Remark 1. A rule on the separable domain satisfies strategy-proofness and no-vetoer if and only if it is voting by no-vetoer committees.

Because of Theorem 1, to show Remark 1, we only need to check that (i) voting by no-vetoer committees is certain to satisfy *no-vetoer*, and (ii) if a rule is voting by committees but not voting by no-vetoer committees, it violates *no-vetoer*. To see (i), by condition (1) of the sets of no-vetoer winning coalitions, any agent i solely does not have veto power against an alternative $z \in K$ with $z_e = 0$. Similarly, by condition (2), any agent i solely cannot veto an alternative z with $z_e = 1$. To see (ii), if condition (1) for some $e \in K$ is violated, then an agent i has a veto power to an alternative z with $z_e = 0$. Similarly, if condition (2) is violated, an agent i has veto power against an alternative z with $z_e = 1$.

[7] Barberà et al. (1991) note that Theorem 1 holds even on the **additive domain**, which is the domain of preferences with additive numerical representations and is strictly smaller than the separable domain, without any technical difficulty. See Barberà et al. (1991) for the precise definition of the additive preferences.

3 The Main Result

In this section, we first define the precise concept of the "maximal domain" following Ching and Serizawa (1998), and then derive the main result.

Maximal domain $\mathcal{D}_M^n \subseteq \mathcal{D}^n$ **for a list of properties.** (1) There exists a rule on $\mathcal{D}_M^n$ satisfying the properties, and (2) for any domain $\mathcal{D}^n$ such that $\mathcal{D}_M \subsetneq \mathcal{D} \subseteq \mathcal{D}_U$, no rule on $\mathcal{D}^n$ satisfies the same properties.

Note that, given a list of properties, there is a possibility that multiple maximal domains exist. Now we can state the main theorem of this paper.

Theorem 2. *The separable domain is a maximal domain for strategy-proofness and no-vetoer.*[8]

Note that there could be another maximal domain that does not contain the separable domain for these properties. However, because separability in preferences is quite important and considered in almost all articles studying this model, this result is at least one of the most interesting maximal domain results for this model.

The proof for this theorem is decomposed into three major steps. As the proof consists of several lemmas and substeps to maintain the generality and is complicated, we move it to the Appendix. Here we provide the proof for the case of $k = 2$ and $n = 3$, which brings basic insight of the general proof. After the proof, we briefly explain the relationship between proofs of this and the general case.

First, we introduce a remark that plays an important role in the proof.

Remark 2. Let $\mathcal{D}_S \subsetneq \mathcal{D} \subseteq \mathcal{D}_U$. Suppose that a rule $f : \mathcal{D}^n \to Z$ satisfies strategy-proofness and no-vetoer. Then there exists voting by no-vetoer committees g such that for all $P \in \mathcal{D}_S^n$, $f(P) = g(P)$.

We obtain this remark immediately from Remark 1 and that f restricted on $\mathcal{D}_S^n$ must satisfy *strategy-proofness* and *no-vetoer*.

When a specific preference P_0 is given beforehand, P_i is employed to denote agent i's preference such that $P_i = P_0$ unless mentioned otherwise. Given $x \in Z$, let $P_0^x \in \mathcal{D}_S$ be such that $\tau(P_0^x) = x$.

Proof of Theorem 2 ($k = 2$ and $n = 3$ Case). Let $\mathcal{D}$ be such that $\mathcal{D}_S \subsetneq \mathcal{D} \subseteq \mathcal{D}_U$. Suppose, on the contrary, that there is a rule f on $\mathcal{D}^3$ satisfying *strategy-proofness* and *no-vetoer*. We derive a contradiction.

Let $\hat{P}_0 \in \mathcal{D} \backslash \mathcal{D}_S$. Without loss of generality, let $\tau(\hat{P}_0) = (1,1) \equiv \tau$ and $z \equiv (0,0)\, \hat{P}_0\, (1,0) \equiv y$. By Remark 2, there exists voting by no-vetoer committees g on $\mathcal{D}_S^3$ such that for each $P \in \mathcal{D}_S^3$, $g(P) = f(P)$. Let $\mathcal{W}$ be the winning coalition system associated with g. Note that by *no-vetoer*, $\mathcal{W}_1 = \mathcal{W}_2 = \{I \subseteq N : \#I = 2\}$. Given $x \in Z$, pick a preference $P_0^x \in \mathcal{D}_S$ such that $\tau(P_0^x) = x$. We also assume that $\tau\, P_0^y\, z$.

[8] By the same proof for Theorem 2, we immediately obtain the result that the separable domain is a unique maximal domain including the additive domain for *strategy-proofness* and *no-vetoer*.

Step 1. Note that $f(P^{\tau}_{\{1,2\}}, P^z_3) = g(P^{\tau}_{\{1,2\}}, P^z_3) = \tau$. Thus, by *strategy-proofness*, we have (1) $f(\hat{P}_1, P^{\tau}_2, P^z_3) = \tau$.
Step 2. Note that $f(P^z_1, P^y_2, P^z_3) = g(P^z_1, P^y_2, P^z_3) = z$ and $f(P^{\tau}_1, P^y_2, P^z_3) = g(P^{\tau}_1, P^y_2, P^z_3) = y$. Consider the outcome of $f(\hat{P}_1, P^y_2, P^z_3)$. If $f(\hat{P}_1, P^y_2, P^z_3) = y$, then agent 1 manipulates f at $(\hat{P}_1, P^y_2, P^z_3)$ via P^z_1, contradicting *strategy-proofness*. If $f(\hat{P}_1, P^y_2, P^z_3) = \tau$, then agent 1 manipulates f at $(P^{\tau}_1, P^y_2, P^z_3)$ via $\hat{P}_1$, contradicting *strategy-proofness*. Therefore, we have that (2) $f(\hat{P}_1, P^y_2, P^z_3) = Z\backslash\{y, \tau\}$.
Step 3. By (1) and (2), agent 2 manipulates f at $(\hat{P}_1, P^y_2, P^z_3)$ via P^{τ}_2, contradicting *strategy-proofness*. □

The proof of the general case has the same structure as that of the special case above. In the proof of the general case, we also fix a domain $\mathcal{D}^n$ such that $\mathcal{D}_S \subsetneq \mathcal{D} \subseteq \mathcal{D}_U$ and take $\hat{P}_0 \in \mathcal{D}\backslash\mathcal{D}_S$. Then we can take P^y_0 and P^z_0 that have the same roles as those in the above simple proof. By two major steps similar to Steps 1 and 2 above, an agent with P^y_0 finally has incentive for manipulation, and we obtain a contradiction.

Finally in this section, we present an example illustrating that *no-vetoer* is indispensable for the theorem. This example shows that a maximal domain for *strategy-proofness*, *ontoness*, and *no-dictator* that includes the separable domain is strictly larger than the separable domain.

Example 1. Let $N = \{1, 2, 3\}$ and $K = \{1, 2\}$. Let $\hat{P}_0 \in \mathcal{D}_U$ be such that $(0, 0)\ \hat{P}_0\ (0, 1)\ \hat{P}_0\ (1, 1)\ \hat{P}_0\ (1, 0)$, and $\mathcal{D} = \mathcal{D}_S \cup \{\hat{P}_0\}$. Let $\mathcal{W}_1 = \{N\}$ and $\mathcal{W}_2 = \{I \subseteq N : \#I \geq 1\}$. Let $f : \mathcal{D}^3 \to Z$ be the voting by committees generated by $\mathcal{W}$. Then f satisfies *strategy-proofness*, *ontoness*, and *no-dictator* but does not satisfy *no-vetoer*.

By the structure of $\mathcal{W}_1$, any agent can be a vetoer against alternatives with object 1 chosen. *No-dictator* is obviously satisfied. Since f is voting by committees, by Theorem 1, *ontoness* is satisfied and no agent with a separable preference has an incentive to misrepresent her preference. If the preference of some agent, say agent i, is $\hat{P}_0$ and she represents her true preference, then by the structure of $\mathcal{W}_1$, the outcome is $(0, 0)$ or $(0, 1)$. In the case of $(0, 0)$, which is agent i's top alternative $\tau(\hat{P}_0)$, it is certain that she has no incentive for misrepresentation. In the case of $(0, 1)$, by $\mathcal{W}$, it follows that the top alternative of one of the other two agents is $(0, 1)$ or $(1, 1)$. Then the outcome that agent i can obtain by misrepresenting her preference is either $(0, 1)$ or $(1, 1)$. Since she prefers $(0, 1)$ to $(1, 1)$, she has no incentive for misrepresentation. Hence f satisfies *strategy-proofness*.

4 Concluding Remarks

In this paper, we have established that *the separable domain is a maximal domain for the properties of strategy-proofness and no-vetoer*. We conclude the article by discussing three topics relating to our result.

The first topic is a question on the uniqueness of maximal domains. Our result does not exclude the possibility that there are other interesting maximal domains for the same properties. When we model a situation, we make assumptions on preferences that are suitable for it. Unless domains include a minimal variety of natural preferences, the results on the domains cannot be applied to interesting situations and become meaningless. Although generally maximal domains are not unique, a maximal domain including small and natural subdomains may be unique. For instance, Barberà et al. (1991) show the uniqueness of a maximal domain that includes a subdomain, called a "minimally rich domain" [9] and on which voting by no-vetoer committees satisfies *strategy-proofness*.[10] A domain $\mathcal{D}^n$ is **minimally rich** if for any $z \in Z$, there is a unique $P_0 \in \mathcal{D}$ such that $\tau(P_0) = z$. In the model where the set of alternatives is a continuous line, without restricting the class of rules a priori, Berga and Serizawa (2000) show the uniqueness of a maximal domain including a minimally rich domain for *strategy-proofness* and *no-vetoer*. Therefore, the following is an interesting open question: *is the separable domain a unique maximal domain including a minimally rich domain for strategy-proofness and no-vetoer?*

The previous studies that obtain unique maximal domains without restricting the class of rules a priori employ characterization results of rules satisfying lists of properties on subdomains. For instance, in establishing the uniqueness of maximal domains, Berga and Serizawa (2000) employ the fact that on a minimally rich domain, the class of rules called "generalized median voter schemes" is a unique class of rules satisfying *strategy-proofness* and *ontoness*. Accordingly, to establish the uniqueness of a maximal domain in the multi-object choice model, it is important whether or not the class of voting by committees is the unique class of rules for *strategy-proofness* on a minimally rich domain. However, as Example 2 below illustrates, *strategy-proof* rules on a minimally rich domain are not necessarily voting by committees.[11] Thus, we need to develop new proof techniques to solve the above open question.

Example 2. Let $N = \{1, 2\}$ and $K = \{1, 2\}$. Let the preferences P_0^A, P_0^B, P_0^C, and P_0^D be such that

$$\begin{aligned} &(1,1)\, P_0^A\, (1,0)\, P_0^A\, (0,1)\, P_0^A\, (0,0), \\ &(1,0)\, P_0^B\, (1,1)\, P_0^B\, (0,0)\, P_0^B\, (0,1), \\ &(0,1)\, P_0^C\, (0,0)\, P_0^C\, (1,1)\, P_0^C\, (1,0), \\ \text{and } &(0,0)\, P_0^D\, (0,1)\, P_0^D\, (0,1)\, P_0^D\, (1,1). \end{aligned}$$

Let $\mathcal{D} = \{P_0^A, P_0^B, P_0^C, P_0^D\}$. Then, $\mathcal{D}^2$ is a minimally rich domain. Consider the rule f as defined by the table below:

	P_2^A	P_2^B	P_2^C	P_2^D
P_1^A	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
P_1^B	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$
P_1^C	$(1,1)$	$(1,1)$	$(0,1)$	$(0,0)$
P_1^D	$(1,1)$	$(1,0)$	$(0,0)$	$(0,0)$

[9] Barberà et al. (1991) refer to a "minimally rich domain" as just a "rich domain".

[10] This comes from Theorem 3 in Barberà et al. (1991).

[11] However, Example 2 does not exclude the possibility that rules satisfying *strategy-proofness* and *no-vetoer* on a minimally rich domain are necessarily voting by committees.

where rows and columns denote the preferences of agents 1 and 2 respectively, and the cells denote the outcomes for the corresponding preference profiles.

If f were voting by committees, then $f(P_1^A, P_2^D) = (1,1)$ and $f(P_1^D, P_2^A) = (1,1)$ imply that the associated classes of winning coalitions are $\mathcal{W}_1 = \mathcal{W}_2 = \{W \subseteq N : \#W \geq 1\}$. This contradicts $f(P_1^C, P_2^D) = f(P_1^C, P_2^D) = (0,0)$. Thus, f is not voting by committees. However, it satisfies *strategy-proofness* and *ontoness*.

The second topic is a question on the class of rules satisfying *strategy-proofness* and *ontoness*. The characterization of such a class is an important theme and is investigated in various models and by many authors. Example 2 above illustrates that a domain smaller than the separable one admits rules that are not voting by committees. A question remains to be answered is what will happen on larger domains. Although Theorem 1 implies that the restrictions of such rules to the separable domain are voting by committees, it does not specify how such rules choose outcomes for nonseparable preferences. In other words, the following question is still open: *is there a rule satisfying strategy-proofness and ontoness on domains larger than the separable domain other than voting by committees?* The merit of our result is that we can obtain a maximal domain without knowing the answer to this question once *ontoness* is strengthened to *no-vetoer*. However, as we discussed in the first topic, the characterizations of rules and maximal domains are closely related, and once we can answer the above question, it might help us to obtain maximal domain results stronger than ours.

The third topic is on "tops-only" property of rules. **Tops-onlyness** states that a rule uses only tops of preference profile to derive the outcome. Chatterji and Sen (2011) establish a strong result that if a domain $\mathcal{D}$ satisfies "Property T^*" defined below, any rule satisfying *strategy-proofness* and "unanimity"[12] on $\mathcal{D}$ is tops-only. A domain $\mathcal{D}$ satisfies **Property** T^* if for each $P_i \in \mathcal{D}$, each $a \in Z \backslash \{\tau(P_i)\}$, and each $x \in Z$ that is preferred to a for each preference in $\mathcal{D}$ whose top is $\tau(P_i)$[13], there exists $\overline{P}_i \in \mathcal{D}$ such that (i) $a = \tau(\overline{P}_i)$ and (ii) for each $y \in Z$ such that $a \, P_i \, y$, $x \, \overline{P}_i \, y$. If Chatterji and Sen's (2011) result could be applied to domains including nonseparable preferences, the results of Barberà et al. (1991) would imply our maximal domain result. However, as Example 3 below illustrates, it cannot be applied to domains including some nonseparable preferences.

Example 3. Let $k = 2$. Let $\hat{P}_0 \in \mathcal{D}$ be a nonseparable preference such that $(1,1) \, \hat{P}_0 \, (0,0) \, \hat{P}_0 \, (1,0) \, \hat{P}_0 \, (0,1)$. Let $\mathcal{D} \equiv \mathcal{D}_S \cup \{\hat{P}_0\}$. Then, this domain $\mathcal{D}$ does not satisfy *Property* T^*. To see that, pick up $\hat{P}_0$ as P_i, and let $a \equiv (0,0)$. Then, only $(1,1)$ is preferred to a for each preference in $\mathcal{D}$ whose top is $\tau(P_i)$. Let $x = (1,1)$. We show that there is no $\overline{P}_i \in \mathcal{D}$ satisfying (i) and (ii) of *Property* T^* for P_i, a and x. Suppose such $\overline{P}_i \in \mathcal{D}$ exists. Then, (i) implies $\overline{P}_i \in \mathcal{D}_S$. Let $y = (1,0)$. Then, $a \, \hat{P}_0 \, y$, but by $\overline{P}_i \in \mathcal{D}_S$ and $\tau(\overline{P}_i) = (0,0), y \, \overline{P}_i \, x$. This contradicts (ii).

Because our result covers the domain $\mathcal{D}$ in Example 3, our result is independent of Chatterji and Sen (2011) and Barberà et al. (1991). However, the following is an important open question: *is there a domain that includes all separable preferences and that is not a tops-only domain, i.e.,*

[12]**Unanimity** requires that if tops of all agents' preferences are the same, then it should be the outcome.
[13]To be precise, for each $P_i' \in \mathcal{D}$ with $\tau(P_i') = \tau(P_i)$, $x \, P_i' \, a$.

a domain, on which there is a rule satisfying strategy-proofness and unanimity but not tops-onlyness?

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Appendix

In this Appendix, the proof of Theorem 2 is provided.

Let $\mathcal{D}_S \subsetneq \mathcal{D} \subset \mathcal{D}_U$. Suppose, on the contrary, that there is a rule f on $\mathcal{D}^n$, satisfying *strategy-proofness* and *no-vetoer*. We derive a contradiction. Let $\hat{P}_0 \in \mathcal{D}\backslash\mathcal{D}_S$. Let $\tau \equiv \tau(\hat{P}_0)$.

Let $A \equiv \{(y,z) \in Z^2 : y \in B(z,\tau) \text{ and } z\,\hat{P}_0\,y\}$. Let $A^* \equiv \{(y,z) \in A : \text{for all } (y',z') \in A, ||z-\tau|| \leq ||z'-\tau||\}$. A is the set of pairs for which $\hat{P}_0$ violates the condition of separability. A^* is the set of pairs in A for which the distances between z and τ are minimal. By $\hat{P}_0 \in \mathcal{D}\backslash\mathcal{D}_S$, $A \neq \varnothing$, and so $A^* \neq \varnothing$. We have the following lemma.

Lemma 1. There exists $(y,z) \in A^*$ such that $||z-y|| = 1$.

Lemma 1 is relatively straightforward and the proof is available in the supplementary note.[14] Hereafter, let $(y,z) \in A^*$ be such that $||z-y|| = 1$. By relabeling coordinates, we have

$$\tau \equiv (1,\cdots\cdots\cdots,1), \quad y \equiv (1,\underbrace{0,\cdots,0}_{a-1},1,\cdots,1), \quad z \equiv (\underbrace{0,0,\cdots,0}_{a},1,\cdots,1),$$

where $a \in K$ is such that $2 \leq a \leq k$.

Given $b \in K$ such that $1 \leq b \leq a$, let $x^b \equiv (\underbrace{1,\cdots,1}_{b},\underbrace{0,\cdots,0}_{a-b},1,\cdots,1)$. Note that $x^1 = y$ and $x^a = \tau$. Also note that since $(y,z) \in A^*$ and $||z-y|| = 1$, $\tau\,\hat{P}_0\,x^{a-1}\,\hat{P}_0\,\cdots\,\hat{P}_0\,x^2\,\hat{P}_0\,y$.

Let $E \equiv Z\backslash B(z,\tau)$. Since $\tau \equiv (1,\cdots,1)$ and $z \equiv (\underbrace{0,\cdots,0}_{a},1,\cdots,1)$,

$$E = \{x \in Z : \text{for some } e \in \{a+1,\cdots,k\}, x_e = 0\}.$$

[14] The supplementary note is attached to the discussion paper version of this study (Hatsumi et al., 2013).

Given $x \in B(z,\tau)$, let $B_x^+ \equiv \{x' \in B(z,\tau) : x' \,\hat{P}_0\, x\}$ and $B_x^- \equiv \{x' \in B(z,\tau) : x \,\hat{P}_0\, x'\}$.

Given $x \in Z$, let $P_0^x \in \mathcal{D}_S$ be such that $\tau(P_0^x) = x$. Assume that for all $x \in B(z,\tau)$ and all $x' \in E$, $x \,P_0^z\, x'$. Assume that for all $x \in B(z,\tau)$ and all $x' \in E$, $x \,P_0^y\, x'$, and for all $w \in B(z,\tau)$ such that $w_1 = 1$ and all $w' \in B(z,\tau)$ such that $w_1' = 0$, $w \,P_0^y\, w'$. Assume that for all $x \in B(z,\tau)$ and all $x' \in E$, $x \,P_0^\tau\, x'$, and P_0^τ and $\hat{P}_0$ are equivalent over $B(z,\tau)\backslash\{z\}$.[15]

By Remark 2, there exists a voting by no-vetoer committees $g : \mathcal{D}_S^n \to Z$ such that for all $P \in \mathcal{D}_S^n$, $f(P) = g(P)$. Let $\mathcal{W}$ be the no-vetoer winning coalition system associated with g.

The next two lemmas are frequently used in the following investigation.

By relabeling agents, we have $I_1 \equiv \{1, \cdots, q\} \in \underline{\mathcal{W}}_1$. Note that by condition (2) of the no-vetoer winning coalition, $2 \leq q \leq n-1$. Given $e \in \{2, \cdots, a\}$, let r_e be such that $I_e \equiv \{1, \cdots, r_e\} \in \mathcal{W}_e$ and $I_e\backslash\{r_e\} \notin \mathcal{W}_e$. By relabeling coordinates, we have $I_2 \subseteq \cdots \subseteq I_a$. Then by condition (2) of no-vetoer winning coalitions, $2 \leq r_2 \leq \cdots \leq r_a \leq n-1$.

Lemma 2. Let $e \in \{2, \cdots, a\}$ and $s \leq r_e - 1$. Let $P \in \mathcal{D}^n$ be such that for all $i \leq s$, $P_i \in \{\hat{P}_0, P_0^\tau\}$, and for all $i \geq s+1$, $P_i \in \{P_0^y, P_0^z\}$. Let $x \equiv f(P)$. Then for all $l \in \{e, \cdots, a\}$, $x_l = 0$.

Proof of Lemma 2. Suppose, on the contrary, that there exists $l \in \{e, \cdots, a\}$ such that $x_l = 1$. Since $f(P) = x$, by the repeated use of *strategy-proofness*, $f(P^x_{\{1,\cdots,s\}}, P_{\{s+1,\cdots,n\}}) = x$. Since $(P^x_{\{1,\cdots,s\}}, P_{\{s+1,\cdots,n\}}) \in \mathcal{D}_S^n$, we have

$$g(P^x_{\{1,\cdots,s\}}, P_{\{s+1,\cdots,n\}}) = f(P^x_{\{1,\cdots,s\}}, P_{\{s+1,\cdots,n\}}) = x.$$

Since $s \leq r_e - 1 \leq r_l - 1$, $\{1, \cdots, s\} \notin \mathcal{W}_l$. This contradicts $g(P^x_{\{1,\cdots,s\}}, P_{\{s+1,\cdots,n\}}) = x$ and $x_l = 1$. □

Lemma 3. Let $j \in N$, $P_{-j} \in \mathcal{D}^{n-1}$, and $x \in B(z,\tau)$. Suppose that $f(P_j^\tau, P_{-j}) = x$. Then (i) $f(\hat{P}_j, P_{-j}) = x$, or (ii) $f(\hat{P}_j, P_{-j}) \in E$ and $f(\hat{P}_j, P_{-j}) \,\hat{P}_0\, x$.

Proof of Lemma 3. Note that $Z = \{x\} \cup B_x^+ \cup B_x^- \cup E$.

If $f(\hat{P}_j, P_{-j}) \in B_x^-$, then agent j manipulates f at $(\hat{P}_j, P_{-j})$ via P_j^τ, contradicting *strategy-proofness.*

Suppose that $f(\hat{P}_j, P_{-j}) \in B_x^+$. Since $\hat{P}_0$ and P_0^τ are equivalent on $B(z,\tau)\backslash\{z\}$, and since $x \,\hat{P}_0\, z$ implies $B_x^+ \subsetneq B(z,\tau)\backslash\{z\}$, $\hat{P}_0$ and P_0^τ are equivalent on B_x^+. Thus, $f(\hat{P}_j, P_{-j}) \in B_x^+$ implies that j manipulates f at (P_j^τ, P_{-j}) via $\hat{P}_j$. This contradicts *strategy-proofness.*

Hence, $f(\hat{P}_j, P_{-j}) = x$ or $f(\hat{P}_j, P_{-j}) \in E$. In the latter case, by *strategy-proofness*, $f(\hat{P}_j, P_{-j}) \,\hat{P}_0\, x$. □

Let $c \in \{2, \cdots, a\}$ be such that $x^c \,\hat{P}_0\, z$ and $z \,\hat{P}_0\, x^{c-1}$. Let d be the maximal element of $\{c, \cdots, a\}$ such that $I_d = I_c$. Let $r \equiv r_c (= r_d)$ and $x^* \equiv x^d$. Note that if $I_c = I_a$, *i.e.*, if $r_c = r_a$, then $d = a$ and $x^* = \tau$, and that if $I_c \subsetneq I_a$, *i.e.*, if $r_c < r_a$, then $d < a$, $x^* \neq \tau$, $I_d \subsetneq I_{d+1}$, and $r_d < r_{d+1}$. Note that $x^* \,\hat{P}_0\, x^c$ or $x^* = x^c$. Then by transitivity, $x^* \,\hat{P}_0\, z$.

[15]Given $Z' \subseteq Z$ and $P_0 \in \mathcal{D}$, let $\tau(P_0, Z') \in Z'$ be such that for all $x \in Z'\backslash\{\tau(P_0, Z')\}$, $\tau(P_0, Z') \; P_0 \; x$. P_0 is **separable over** Z' if for all $y', z' \in Z'\backslash\{\tau(P_0, Z')\}$, $y' \neq z'$ and $y' \in B(z', \tau(P_0))$ imply $y' \,P_0\, z'$. Note that since $(y,z) \in A^*$, $\hat{P}_0$ satisfies separability over $B(z,\tau)\backslash\{z\}$.

There are two cases, A and B. Case A is that $I_1 \subseteq I_d$, *i.e.*, $q \leq r$. Case B is that $I_d \subsetneq I_1$, *i.e.*, $r < q$. We derive a contradiction in each of the two cases.

Case A. ($I_1 \subseteq I_d$, *i.e.*, $q \leq r$.)

Step 1. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) = x^*$.

We add a lemma and then prove this step.

Lemma 4. Let $0 \leq j \leq r-2$. Let $f(\hat{P}_{\{1,\cdots,j\}}, P^\tau_{\{j+1,\cdots,r\}}, P^z_{-I_d}) \in E$, and $f(\hat{P}_{\{1,\cdots,j\}}, P^\tau_{\{j+1,\cdots,r\}}, P^z_{-I_d})\, \hat{P}_0\, x^*$. Then $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^\tau_{\{j+2,\cdots,r\}}, P^z_{-I_d}) \in E$ and $x\, \hat{P}_0\, x^*$.

Proof of Lemma 4. Since $f(\hat{P}_{\{1,\cdots,j\}}, P^\tau_{\{j+1,\cdots,r\}}, P^z_{-I_d})\, \hat{P}_0\, x^*$, $d < a$, and *strategy-proofness*, $x\, \hat{P}_0\, x^*$.

Suppose that $x \notin E$, *i.e.*, $x \in B(z,\tau)$. By $x\, \hat{P}_0\, x^*$ and $x^*\, \hat{P}_0\, z$, $x\, \hat{P}_0\, z$ and so $x \neq z$. By $x \neq z$, $x \in B(z,\tau)\backslash\{z\}$. Then since $\hat{P}_0$ satisfies separability on $B(z,\tau)\backslash\{z\}$ and $x\, \hat{P}_0\, x^*$, $x^* \notin B(x,\tau)$. Since $\tau \equiv (1,\cdots\cdots\cdots,1)$ and $x^* = (\underbrace{1,\cdots,1}_{d},\underbrace{0,\cdots,0}_{a-d},1,\cdots,1)$, and since $x \in B(z,\tau)$ imply that for all $e \in \{a+1,\cdots,n\}$, $x_e = 1$, it follows that for some $e \in \{d+1,\cdots,a\}$, $x_e = 1$. On the other hand, since $d+1 \leq e$ implies $r_{d+1} \leq r_e$, $r = r_d < r_{d+1}$ implies $r \leq r_e - 1$. Thus, by Lemma 2, $x_e = 0$. This is a contradiction. Hence, $x \in E$. □

Proof of Step 1. Suppose that $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) \neq x^*$. We derive a contradiction in three substeps.

Substep 1-1. Since $(P^\tau_{I_d}, P^z_{-I_d}) \in \mathcal{D}^n_S$, $I_d \in \mathcal{W}_e$ for all $e \in \{1,\cdots,d\}$, and $I_d \notin \mathcal{W}_e$ for all $e \in \{d+1,\cdots,a\}$, we have $f(P^\tau_{I_d}, P^z_{-I_d}) = g(P^\tau_{I_d}, P^z_{-I_d}) = x^*$. By first applying Lemma 3, and then $r-2$ additional times either Lemma 3 or Lemma 4, we obtain that (i) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) = x^*$, or (ii) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) \in E$ and $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d})\, \hat{P}_0\, x^*$. Since we suppose that $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) \neq x^*$, we have $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) \in E$ and $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d})\, \hat{P}_0\, x^*$. Note that if $d = a$, then $x^* = \tau$. This contradicts $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d})\, \hat{P}_0\, x^*$. Thus $d < a$ and $x^* \neq \tau$.

Substep 1-2. Let d' be the maximal element of $\{d+1,\cdots,a\}$ such that $I_{d'} = I_{d+1}$. Let $r' \equiv r_{d'}$ and $x' \equiv x^{d'}$. Note that if $I_{d+1} = I_a$, *i.e.*, if $r_{d+1} = r_a$, then $d' = a$ and $x' = \tau$, and that if $I_{d+1} \subsetneq I_a$, *i.e.*, if $r_{d+1} < r_a$, then $d' < a$, $x' \neq \tau$ and $r_{d'} < r_{d'+1}$. In this substep, we show that (i) $f(P^\tau_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = x'$, (ii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}})\, \hat{P}_0\, x'$, and (iii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \in E$.

Since $(P^\tau_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \in \mathcal{D}^n_S$, $I_{d'} = \{1,\cdots,r'\} \in \mathcal{W}_e$ for all $e \in \{1,\ldots,d'\}$ and $I_{d'} \notin \mathcal{W}_e$ for all $e \in \{d'+1,\ldots,a\}$, $f(P^\tau_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = g(P^\tau_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = x'$. Thus, we have (i).

Let $x \equiv f(\hat{P}_{\{1,\cdots,r'-1\}}, P^z_{\{r',\cdots,n\}})$. In this paragraph, we show $x \in E$. Since $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d})\, \hat{P}_0\, x^*$, by the repeated use of *strategy-proofness*, $x\, \hat{P}_0\, x^*$. Thus $x \in B^+_{x^*} \cup E$. Suppose that $x \in B^+_{x^*}$. Since $\hat{P}_0$ satisfies separability on $B^+_{x^*}$, $\tau \equiv (1,\cdots\cdots\cdots,1)$, and $x^* = (\underbrace{1,\cdots,1}_{d},\underbrace{0,\cdots,0}_{a-d},1,\cdots,1)$, $x \in B^+_{x^*}$ implies that for some $e \in \{d+1,\cdots,a\}$, $x_e = 1$. Let $e \in \{d+1,\cdots,a\}$ be such that $x_e = 1$. By $d+1 \leq e$, $r_{d+1} \leq r_e$. Thus, $r' \equiv r_{d'} = r_{d+1}$ implies $r'-1 \leq r_e - 1$. Accordingly, by Lemma 2, $x_e = 0$. This is a contradiction. Therefore, $x \in E$.

Let $y' \equiv f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}})$. If $y' = x'$, then by $x' \in B(z,\tau)$ and the definition of P^z_0, for all $z' \in E$, $y' \, P^z_0 \, z'$. By $x \in E$, this implies that agent r' manipulates f at $(\hat{P}_{\{1,\cdots,r'-1\}}, P^z_{\{r',\cdots,n\}})$ via $\hat{P}_{r'}$. This contradicts *strategy-proofness*. Thus, $y' \neq x'$. By $y' \neq x'$, and the repeated use of *strategy-proofness* to (i), we have (ii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = y' \, \hat{P}_0 \, x'$.

Suppose that $y' \notin E$. Then, by $y' \neq x'$, $y' \in B(z,\tau)\backslash\{x'\}$. By $y' \, \hat{P}_0 \, x'$, $x' \neq \tau$ and so $d' < a$. Note that since $x' \, \hat{P}_0 \, x^*$ and $x^* \, \hat{P}_0 \, z$, $B^+_{x'} \cup \{x'\} \subseteq B(z,\tau)\backslash\{z\}$. Since $\hat{P}_0$ is separable on $B(z,\tau)\backslash\{z\}$, it is separable on $B^+_{x'} \cup \{x'\}$. Thus by $y' \, \hat{P}_0 \, x'$, $x' \notin B(y',\tau)$. Then since $\tau \equiv (1, \cdots\cdots\cdots, 1)$, and $x' = (\underbrace{1, \cdots, 1}_{d'}, \underbrace{0, \cdots, 0}_{a-d'}, 1, \cdots, 1)$, for some $e \in \{d'+1, \cdots, a\}$, we have $y'_e = 1$. By $d'+1 \leq e$, $r_{d'+1} \leq r_e$. Thus, $r' \equiv r_{d'} < r_{d'+1} \leq r_e$, and so $r' \leq r_e - 1$. Therefore, by Lemma 2, $y'_e = 0$. This is a contradiction. Thus, $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = y' \in E$.

Substep 1-3. As we show in Substep 1-2, $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \, \hat{P}_0 \, x'$ and $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \in E$. Similarly to Substep 1-1, we have $d' < a$. Let d'' be the maximal element of $\{d'+1, \cdots, a\}$ such that $I_{d''} = I_{d'+1}$. Let $r'' \equiv r_{d''}$ and $x'' \equiv x^{d''}$. Then we can repeat the argument of Substep 1-2 by replacing r' with r'', x^* with x' and x'' with x'. As a result, we obtain that $f(\hat{P}_{\{1,\cdots,r''\}}, P^z_{\{r''+1,\cdots,n\}}) \, \hat{P}_0 \, x''$ and $f(\hat{P}_{\{1,\cdots,r''\}}, P^z_{\{r''+1,\cdots,n\}}) \in E$.

Repeat the argument. Then, finally, we have that $f(\hat{P}_{\{1,\cdots,r_a-1\}}, P^z_{\{r_a,\cdots,n\}}) \in E$. Note that $f(P^\tau_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = g(P^\tau_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = \tau$. Thus by the repeated use of *strategy-proofness*, $f(\hat{P}_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = \tau$. Then agent r_a manipulates f at $(\hat{P}_{\{1,\cdots,r_a-1\}}, P^z_{\{r_a,\cdots,n\}})$ via $\hat{P}_{r_a}$. This contradicts *strategy-proofness*. Hence, we have $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) = x^*$. □

Step 2. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = z$ or $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \in E$.

We add two lemmas, and then prove this step.

Lemma 5. Let $0 \leq j \leq r-2$. Let $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = z$. Then
(i) $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = z$, or
(ii) $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \in E$ and $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \, \hat{P}_0 \, z$.

Proof of Lemma 5. Note that $Z = \{z\} \cup B^+_z \cup B^-_z \cup E$. Let $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d})$.

(I) If $x \in B^-_z$, then agent $j+1$ manipulates f at $(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d})$ via P^z_{j+1}, which contradicts *strategy-proofness*.

(II) Suppose that $x \in B^+_z$. Then by the repeated use of *strategy-proofness*, $f(P^x_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = x$. Since $x \in B^+_z$ and $z \, \hat{P}_0 \, x^{c-1}$, we have $x \neq z$, $x^{c-1} \neq z$, and $x \, \hat{P}_0 \, x^{c-1}$. Then since $\hat{P}_0$ satisfies separability on $B(z,\tau)\backslash\{z\}$, $x^{c-1} \notin B(x,\tau)$. Since $\tau \equiv (1, \cdots\cdots\cdots, 1)$, $x^{c-1} = (\underbrace{1, \cdots, 1}_{c-1}, \underbrace{0, \cdots, 0}_{a-(c-1)}, 1, \cdots, 1)$, and $x \in B(z,\tau)$ implies that for all $e \in \{a+1, \cdots, n\}$, $x_e = 1$, it follows that for some $e \in \{c, \cdots, a\}$, $x_e = 1$. Let $e \in \{c, \cdots, a\}$ be such that $x_e = 1$. By $c \leq e$, $r = r_c \leq r_e$, and so $r-1 \leq r_e - 1$. Therefore, by Lemma 2, $x_e = 0$. This is a contradiction.

Hence, we obtain that $x = z$ or $x \in E$. In the latter case, by *strategy-proofness*, $x \, \hat{P}_0 \, z$. □

Lemma 6. Let $1 \leq j \leq r-2$. Suppose that $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \in E$ and $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \, \hat{P}_0 \, z$. Then $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \in E$ and $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) \, \hat{P}_0 \, z$.

Proof of Lemma 6. Let $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_r, P^z_{-I_d})$. Since $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_r, P^z_{-I_d})\, \hat{P}_0\, z$, by *strategy-proofness*, $x\, \hat{P}_0\, z$. Suppose that $x \notin E$. Then by $x\, \hat{P}_0\, z$, $x \in B^+_z$. Then, in the same way as in case (II) of Lemma 5, we obtain a contradiction. □

Proof of Step 2. $f(P^z_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = g(P^z_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_d}) = z$. By first applying Lemma 5, and then $r-2$ additional times either Lemma 5 or Lemma 6, we obtain the statement of this step. □

Step 3. *Proof of Case A.* By Step 1, we have (1) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_2}) = x^*$. By Step 2, we have (2) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_2}) = z$, or (3) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_2}) \in E$. In either (2) or (3), by comparing with (1), agent r manipulates f at $(\hat{P}_{\{1,\cdots,r-1\}}, P^y_r, P^z_{-I_2})$ via P^τ_r, which contradicts *strategy-proofness*. □

Case B. ($I_d \subsetneq I_1$, *i.e.*, $r < q$.)

The argument of Case B is parallel to that of Case A, but for different points. Let h be the maximal element of $\{d, \cdots, a\}$ such that $I_h \subseteq I_1$, *i.e.*, $r_h \leq q$. Let $x^{**} \equiv x^h$. Note that $h = a$ and $x^{**} = \tau$ if and only if $r_a \leq q$, and that $h < a$ and $x^{**} \neq \tau$ if and only if $r_a > q$, and that if $r_a > q$, $r_h \leq q < r_{h+1}$. Also note that $x^{**}\, \hat{P}_0\, x^*$ or $x^{**} = x^*$. Thus $x^{**}\, \hat{P}_0\, z$.

Parallel to Steps 1 and 2, we can show Steps 4 and 5 below. Their precise proofs are available in the supplementary note.[16] Since the proof for Step 6 is slightly different from that for Step 3, we present it here.

Step 4. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_{\{r,\cdots,q\}}, P^z_{-I_1}) = x^{**}$.

Step 5. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$ or $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$.

Step 6. *Proof of Case B.* By Step 4, we have (4) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_{\{r,\cdots,q\}}, P^z_{-I_1}) = x^{**}$. By repeated use of *strategy-proofness* to (4), we have (5) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^y_{\{r+1,\cdots,q\}}, P^z_{-I_1}) \in \{x \in Z : x\, P^y_0\, x^{**}\} \cup \{x^{**}\}$. By Step 5, we have (6) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$, or (7) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$. Note that by the definition of P^y_0, $x^{**}\, P^y_0\, z$ and for all $y' \in E$, $x^{**}\, P^y_0\, y'$. Thus in either (6) or (7), by comparing (5), agent r manipulates f at $(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})$ via P^τ_r, which contradicts *strategy-proofness*. □

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Supplementary note for "A maximal domain for strategy-proof and no-vetoer rules in the multi-object choice model"

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In this supplementary note, we provide the proofs of Lemma 1 and Steps 4 and 5, which are omitted in the main paper.

Lemma 1. There exists $(y, z) \in A^*$ such that $||z - y|| = 1$.

Proof of Lemma 1. Suppose, on the contrary, that for all $(y, z) \in A^*$, $||z - y|| > 1$. Let $(y, z) \in A^*$. By $||z - y|| > 1$, there is $x \in B(z, \tau)$ such that $||z - x|| = 1$, and $y \in B(x, \tau)$. If $z \, \hat{P}_0 \, x$, then $x \in B(z, \tau)$ implies $(x, z) \in A^*$, and so $||z - x|| = 1$ contradicts the hypothesis. If $x \, \hat{P}_0 \, z$, then $z \, \hat{P}_0 \, y$ implies $x \, \hat{P}_0 \, y$, and so $y \in B(x, \tau)$ implies $(y, x) \in A$. Since $x \in B(z, \tau)$ and $||z - x|| = 1$ imply $||x - \tau|| < ||z - \tau||$, this contradicts $(y, z) \in A^*$. □

Step 4. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^z_{-I_1}) = x^{**}$.

We add a lemma and then prove this step.

Lemma 7. Let $1 \leq j \leq r-2$. Let $f(\hat{P}_{\{1,\cdots,j\}}, P^{\tau}_{\{j+1,\cdots,q\}}, P^z_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,j\}}, P^{\tau}_{\{j+1,\cdots,q\}}, P^z_{-I_1}) \, \hat{P}_0 \, x^{**}$. Then $f(\hat{P}_{\{1,\cdots,j+1\}}, P^{\tau}_{\{j+2,\cdots,q\}}, P^z_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,j+1\}}, P^{\tau}_{\{j+2,\cdots,q\}}, P^z_{-I_1}) \, \hat{P}_0 \, x^{**}$.

Proof of Lemma 7. Let $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^{\tau}_{\{j+2,\cdots,q\}}, P^z_{-I_1})$. By $f(\hat{P}_{\{1,\cdots,j\}}, P^{\tau}_{\{j+1,\cdots,q\}}, P^z_{-I_1}) \, \hat{P}_0 \, x^{**}$, $x^{**} \neq \tau$, and by *strategy-proofness*, $x \, \hat{P}_0 \, x^{**}$. By $x^{**} \neq \tau$, $h < a$, and $r_h \leq q < r_{h+1}$.

Suppose that $x \notin E$, *i.e.*, $x \in B(z, \tau)$. By $x \, \hat{P}_0 \, x^{**}$ and $x^{**} \, \hat{P}_0 \, z$, $x \, \hat{P}_0 \, z$ and so $x \neq z$. By $x \neq z$, $x \in B(z, \tau) \backslash \{z\}$. Then since $\hat{P}_0$ satisfies separability on $B(z, \tau) \backslash \{z\}$ and $x \, \hat{P}_0 \, x^{**}$, $x^{**} \notin B(x, \tau)$. Since

$$\tau \equiv (1, \cdots\cdots\cdots, 1),$$
$$x^{**} = (\underbrace{1, \cdots, 1}_{h}, \underbrace{0, \cdots, 0}_{a-h}, 1, \cdots, 1)$$

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and since $x \in B(z,\tau)$ imply that for all $e \in \{a+1,\cdots,n\}$, $x_e = 1$, it follows that for some $e \in \{h+1,\cdots,a\}$, $x_e = 1$. On the other hand, since $h+1 \leq e$ implies $r_{h+1} \leq r_e$, $q < r_{h+1}$ implies $q \leq r_e - 1$. Thus, by Lemma 2, $x_e = 0$. This is a contradiction. Hence, $x \in E$. □

Proof of Step 4. Suppose that $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1}) \neq x^{**}$. We derive a contradiction in three substeps.

Substep 4-1. Since for all $e \in \{1,\cdots,h\}$, $I_1 \in \mathcal{W}_e$ and for all $e \in \{h+1,\cdots,a\}$,[1] $I_1 \notin \mathcal{W}_e$, we have $f(P^{\tau}_{I_1}, P^{z}_{-I_1}) = g(P^{\tau}_{I_1}, P^{z}_{-I_1}) = x^{**}$. By first applying Lemma 3, and then $r-2$ additional times either Lemma 3 or Step 7, we obtain that (i) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1}) = x^{**}$ or (ii) $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1})\, \hat{P}_0\, x^{**}$. Since we suppose that $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1}) \neq x^{**}$, we have $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1})\, \hat{P}_0\, x^{**}$. Note that if $h = a$, then $x^{**} = \tau$. This contradicts $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1})\, \hat{P}_0\, x^{**}$. Thus $h < a$ and $x^{**} \neq \tau$. By $h < a$, $q < r_a$ and $r_h \leq q < r_{h+1}$.

Substep 4-2. Let h' be the maximal element of $\{h+1,\cdots,a\}$ such that $I_{h'} = I_{h+1}$. Let $r' \equiv r_{h'}$ and $x^{**\prime} \equiv x^{h'}$. Note that $r' > q$, that if $I_{h+1} = I_a$, *i.e.*, if $r_{h+1} = r_a$, then $h' = a$ and $x^{**\prime} = \tau$, and that if $I_{h+1} \subsetneq I_a$, *i.e.*, if $r_{h+1} < r_a$, then $h' < a$, $x^{**\prime} \neq \tau$ and $r_{h'} < r_{h'+1}$. In this substep, we show that (i) $f(P^{\tau}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) = x^{**\prime}$, (ii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}})\, \hat{P}_0\, x^{**\prime}$, and (iii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) \in E$.

Since $(P^{\tau}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) \in \mathcal{D}^n_S$, $I_{h'} = \{1,\cdots,r'\} \in \mathcal{W}_e$ for all $e \in \{1,\ldots,h'\}$ and $I_{h'} \notin \mathcal{W}_e$ for all $e \in \{h'+1,\ldots,a\}$, we have

$$\text{(i) } f(P^{\tau}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) = g(P^{\tau}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) = x^{**\prime}.$$

Let $x \equiv f(\hat{P}_{\{1,\cdots,r'-1\}}, P^{z}_{\{r',\cdots,n\}})$. In this paragraph, we show $x \in E$. Since $f(\hat{P}_{\{1,\cdots,r-1\}}, P^{\tau}_{\{r,\cdots,q\}}, P^{z}_{-I_1})\, \hat{P}_0\, x^{**}$, by the repeated use of *strategy-proofness*, $x\, \hat{P}_0\, x^{**}$. Thus $x \in B^{+}_{x^{**}} \cup E$. Suppose that $x \in B^{+}_{x^{**}}$. Since $\hat{P}_0$ satisfies separability on $B^{+}_{x^{**}}$ and

$$\tau \equiv (1,\cdots\cdots\cdots,1),$$
$$x^{**} = (\underbrace{1,\cdots,1}_{h},\underbrace{0,\cdots,0}_{a-h},1,\cdots,1),$$

$x \in B^{+}_{x^{**}}$ implies that for some $e \in \{h+1,\cdots,a\}$, $x_e = 1$. Let $e \in \{h+1,\cdots,a\}$be such that $x_e = 1$. By $h+1 \leq e$, $r_{h+1} \leq r_e$. Thus, $r' \equiv r_{h'} = r_{h+1}$ implies $r'-1 \leq r_e - 1$. Accordingly, by Lemma 2, $x_e = 0$. This is a contradiction. Therefore, $x \in E$.

Let $y' \equiv f(\hat{P}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}})$. If $y' = x^{**\prime}$, then by $x^{**\prime} \in B(z,\tau)$ and the definition of P^z_0, for all $z' \in E$, $y'\, P^z_0\, z'$. By $x \in E$, this implies that agent r' manipulates f at $(\hat{P}_{\{1,\cdots,r'-1\}}, P^{z}_{\{r',\cdots,n\}})$ via $\hat{P}_{r'}$. This contradicts *strategy-proofness*. Thus, $y' \neq x^{**\prime}$.

By $y' \neq x^{**\prime}$, and the repeated use of *strategy-proofness* to (i), we have (ii) $f(\hat{P}_{\{1,\cdots,r'\}}, P^{z}_{\{r'+1,\cdots,n\}}) = y'\, \hat{P}_0\, x^{**\prime}$.

Suppose that $y' \notin E$. Then, by $y' \neq x^{**\prime}$, $y' \in B(z,\tau)\backslash\{x^{**\prime}\}$. By $y'\, \hat{P}_0\, x^{**\prime}$, $x^{**\prime} \neq \tau$ and so

[1] If $r_a \leq q$, then $h = a$, and so $\{h+1,\cdots,a\} = \varnothing$.

$h' < a$. Note that since $x^{**\prime} \hat{P}_0 x^*$ and $x^* \hat{P}_0 z$, $B^+_{x^{**\prime}} \cup \{x^{**\prime}\} \subseteq B(z,\tau)\backslash\{z\}$. Since $\hat{P}_0$ is separable on $B(z,\tau)\backslash\{z\}$, it is separable on $B^+_{x^{**\prime}} \cup \{x^{**\prime}\}$. Thus by $y' \hat{P}_0 x^{**\prime}$, $x^{**\prime} \notin B(y',\tau)$. Then since

$$\tau \equiv (1,\cdots\cdots\cdots,1),$$
$$x^{**\prime} = (\underbrace{1,\cdots,1}_{h'},\underbrace{0,\cdots,0}_{a-h'},1,\cdots,1),$$

for some $e \in \{h'+1,\cdots,a\}$, $y'_e = 1$. By $h'+1 \leq e$, $r_{h'+1} \leq r_e$. Thus, $r' \equiv r_{h'} < r_{h'+1} \leq r_e$, and so $r' \leq r_e - 1$. Therefore, by Lemma 2, $y'_e = 0$. This is a contradiction. Therefore, $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) = y' \in E$.

Substep 4-3. As we show in Substep 4-2, $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \hat{P}_0 x^{**\prime}$ and $f(\hat{P}_{\{1,\cdots,r'\}}, P^z_{\{r'+1,\cdots,n\}}) \in E$. Similarly to Substep 4-1, we have $h' < a$. Let h'' be the maximal element of $\{h'+1,\cdots,a\}$ such that $I_{h''} = I_{h'+1}$. Let $r'' \equiv r_{h''}$ and $x^{**\prime\prime} \equiv x^{h''}$. Then we can repeat the argument of Substep 4-2 by replacing r' with r'', x^{**} with $x^{**\prime}$ and $x^{**\prime\prime}$ with $x^{**\prime}$. As a result, we obtain that $f(\hat{P}_{\{1,\cdots,r''\}}, P^z_{\{r''+1,\cdots,n\}}) \hat{P}_0 x^{**\prime\prime}$ and $f(\hat{P}_{\{1,\cdots,r''\}}, P^z_{\{r''+1,\cdots,n\}}) \in E$.

Repeat the argument. Then finally, we have that $f(\hat{P}_{\{1,\cdots,r_a-1\}}, P^z_{\{r_a,\cdots,n\}}) \in E$. Note that $f(P^\tau_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = g(P^\tau_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = \tau$. Thus by the repeated use of *strategy-proofness*, $f(\hat{P}_{\{1,\cdots,r_a\}}, P^z_{\{r_a+1,\cdots,n\}}) = \tau$. Then agent r_a manipulates f at $(\hat{P}_{\{1,\cdots,r_a-1\}}, P^z_{\{r_a,\cdots,n\}})$ via $\hat{P}_{r_a}$. This contradicts *strategy-proofness*.

Hence, we have $f(\hat{P}_{\{1,\cdots,r-1\}}, P^\tau_r, P^z_{-I_d}) = x^{**}$ □

Step 5. $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$ or $f(\hat{P}_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$.

We add two lemmas, and then prove this step.

Lemma 8. Let $j \in \{0,\cdots,r-2\}$. Let $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$. Then
(i) $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$, or
(ii) $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \hat{P}_0 z$.

Proof of Lemma 8.

Note that $Z = \{z\} \cup B^+_z \cup B^-_z \cup E$. Let $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})$.

(I) If $x \in B^-_z$, then agent $j+1$ manipulates f at $(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})$ via P^z_{j+1}, which contradicts *strategy-proofness*.

(II) Suppose that $x \in B^+_z$. Then by the repeated use of *strategy-proofness*, $f(P^x_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = x$. Since $x \in B^+_z$ and $z \hat{P}_0 x^{c-1}$, we have: $x \neq z$, $x^{c-1} \neq z$, and $x \hat{P}_0 x^{c-1}$. Then since $\hat{P}_0$ satisfies separability on $B(z,\tau)\backslash\{z\}$, $x^{c-1} \notin B(x,\tau)$. Since

$$\tau \equiv (1,\cdots\cdots\cdots,1),$$
$$x^{c-1} = (\underbrace{1,\cdots,1}_{c-1},\underbrace{0,\cdots,0}_{a-(c-1)},1,\cdots,1)$$

and $x \in B(z,\tau)$ implies that for all $e \in \{a+1,\cdots,n\}$, $x_e = 1$, it follows that for some $e \in \{c,\cdots,a\}$, $x_e = 1$. Let $e \in \{c,\cdots,a\}$ be such that $x_e = 1$. By $c \leq e$, $r = r_c \leq r_e$, and so $r-1 \leq r_e - 1$. Therefore, by Lemma 2, $x_0 = 0$. This is a contradiction.

Hence, we obtain that $x = z$ or $x \in E$. In the latter case, by *strategy-proofness*, $x\,\hat{P}_0\,z$. $\square$

Lemma 9. Let $j \in \{1, \cdots, r-2\}$. Suppose that $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})\,\hat{P}_0\,z$. Then $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) \in E$ and $f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})\,\hat{P}_0\,z$.

Proof of Lemma 9. Let $x \equiv f(\hat{P}_{\{1,\cdots,j+1\}}, P^z_{\{j+2,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})$. Since $f(\hat{P}_{\{1,\cdots,j\}}, P^z_{\{j+1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1})\,\hat{P}_0\,z$, by *strategy-proofness*, $x\,\hat{P}_0\,z$. Suppose that $x \notin E$. Then by $x\,\hat{P}_0\,z$, $x \in B^+_z$. Then, in the same way as in case (II) of Step 8, we obtain a contradiction. $\square$.

Proof of Step 5. By $\{1, \cdots, q\} \in \underline{\mathcal{W}}_1$ and $2 \leq r$, $\{r, \cdots, q\} \notin \mathcal{W}_1$. Thus $f(P^z_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = g(P^z_{\{1,\cdots,r-1\}}, P^y_{\{r,\cdots,q\}}, P^z_{-I_1}) = z$. By first applying Lemma 8, and then $r-2$ additional times either Lemma 8 or Lemma 9, we obtain the statement of this step. $\square$