On the Turaev-Viro-Ocneanu invariant of 3-manifolds derived from generalized E_6-subfactors

Michihisa Wakui*

This is a joint work with Nobuya Sato (Rikkyo University).

At the beginning of the 1990's, a $(2+1)$-dimensional unitary topological quantum field theory, in short, TQFT, was introduced by A. Ocneanu [10] by using a type II_1 subfactor with finite index and finite depth as a generalization of the Turaev-Viro TQFT [12] which was derived from the quantum group $U_q(sl(2,\mathbb{C}))$ at certain roots of unity. We call such a TQFT a Turaev-Viro-Ocneanu TQFT.

When a topological invariant for manifolds is given, it is a fundamental problem to know whether the invariant is determined only by homotopy type of manifolds, or not. It has already known that the Witten-Reshetikhin-Turaev invariant distinguishes the lens spaces $L(7,1)$ and $L(7,2)$, that are orientation preserving homotopic but not homeomorphic. The same problem is open for Turaev-Viro-Ocneanu invariants from subfactors.

In our previous paper [11], we computed Turaev-Viro-Ocneanu invariants from several subfactors for basic 3-manifolds including lens spaces and Brieskorn 3-manifolds. As a result, we showed that $L(p,1)$ and $L(p,2)$ are distinguished by the Turaev-Viro-Ocneanu invariant from a generalized E_6-subfactor with the cyclic group $\mathbb{Z}/p\mathbb{Z}$ for $p=3,5$. From this fact, it is natural for us to expect that the lens spaces $L(7,1)$ and $L(7,2)$ are distinguished by a generalized E_6-subfactor with $\mathbb{Z}/7\mathbb{Z}$. However, at that time, it was not known that there is such a subfactor. Recently, by using sector theory, Izumi [7] found new subfactors including a generalized E_6-subfactor with $\mathbb{Z}/7\mathbb{Z}$. In this note, we report results of computation of Turaev-Viro-Ocneanu invariants from such subfactors for lens spaces $L(p,q)$ in the case where $p \leq 7$ is an odd integer.

For a complex number a, the symbol $\bar{a}$ denotes the complex conjugate of a.

§1. Generalized E_6-subfactors

Generalized E_6-subfactors [6] are new subfactors found by Izumi based on the theory of sectors. In this section, we prepare some terminologies from subfactor theory, and describe the definition of generalized E_6-subfactors.

Let $\mathcal{M}$ be an infinite factor. We denote by $\mathrm{End}_0(\mathcal{M})$ the set of $*$-endomorphisms ρ such that the minimal index $[\mathcal{M} : \rho(\mathcal{M})]_0$ is finite. For $\rho, \eta \in \mathrm{End}_0(\mathcal{M})$ the intertwiner space $\mathrm{Hom}(\rho,\eta)$ is defined by

$$\mathrm{Hom}(\rho,\eta) = \{\ T \in \mathcal{M} \mid T\rho(x) = \eta(x)T \quad \text{for} \quad x \in \mathcal{M}\ \}.$$

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This is a vector space. If $\rho \in \mathrm{End}_0(\mathcal{M})$ is irreducible, namely $\dim \mathrm{Hom}(\rho,\rho)=1$, then for any $\eta \in \mathrm{End}_0(\mathcal{M})$ the intertwiner space $\mathrm{Hom}(\rho,\eta)$ is a Hilbert space with the inner product

$$(T,T') = T^*T' \in \mathrm{Hom}(\rho,\rho) \cong \mathbb{C} \quad \text{for} \quad T,T' \in \mathrm{Hom}(\rho,\eta).$$

Two $*$-endomorphisms $\rho,\eta \in \mathrm{End}_0(\mathcal{M})$ are unitary equivalent if there is an element $U \in \mathcal{M}$ such that $U\rho_1(x) = \rho_2(x)U$ for all $x \in \mathcal{M}$ and $UU^* = U^*U = 1$. The unitary equivalence class $[\rho]$ is called a sector.

For $\rho \in \mathrm{End}_0(\mathcal{M})$ we set $d(\rho) = \sqrt{[\mathcal{M} : \rho(\mathcal{M})]_0}$, and call it the statistical dimension of ρ. It is known that for every $\rho \in \mathrm{End}_0(\mathcal{M})$ there is a $*$-endomorphism $\bar\rho \in \mathrm{End}_0(\mathcal{M})$ and a pair of intertwiners $R_\rho \in \mathrm{Hom}(\mathrm{id},\bar\rho\rho)$, $\overline{R}_\rho \in \mathrm{Hom}(\mathrm{id},\rho\bar\rho)$ such that

$$\overline{R}_\rho^*\rho(R_\rho) = R_\rho^*\bar\rho(\overline{R}_\rho) = \frac{1}{d(\rho)}, \; R_\rho^*R_\rho = \overline{R}_\rho^*\overline{R}_\rho = 1.$$

Such $\bar\rho$ is unique up to unitary equivalence. So we call it the conjugation of ρ.

The set of unitary equivalence classes on $\mathrm{End}_0(\mathcal{M})$ has a structure of $*$-semiring over $\mathbb{C}$, whose product is induced by composition of maps $\rho\eta = \rho\circ\eta$, and whose $*$-action is induced by taking the conjugation [5, 9].

A finite subset Δ of $\mathrm{End}_0(\mathcal{M})$ is called a finite irreducible system closed under sector operations if the following four conditions are satisfied [4].

(i) $\mathrm{id}_{\mathcal{M}} \in \Delta$.

(ii) For all $\rho,\eta \in \Delta$, $\dim \mathrm{Hom}(\rho,\eta) = \begin{cases} 1 & \text{if } \rho=\eta, \\ 0 & \text{otherwise.}\end{cases}$

(iii) For every $\rho \in \Delta$ the conjugation $\bar\rho$ is also in Δ.

(iv) For $\rho,\eta,\zeta \in \Delta$ with $\dim \mathrm{Hom}(\zeta,\rho\eta) \neq 0$, there is an orthonormal basis $\{T_i\}$ in $\mathrm{Hom}(\zeta,\rho\eta)$ such that

$$(*) \qquad \sum_{\zeta\in\Delta}\sum_i T_iT_i^* = 1, \qquad (\rho\eta)(x) = \sum_{\zeta\in\Delta}\sum_i T_i\zeta(x)T_i^* \quad \text{for all } x \in \mathcal{M}.$$

The condition (iv) is equivalent to that there are non-negative integers $N_{\rho\eta}^\zeta$ such that

$$[\rho][\eta] = \bigoplus_{\zeta\in\Delta} N_{\rho\eta}^\zeta[\zeta].$$

Izumi [6] introduced a new class of subfactors as generalizations of E_6-subfactors. They arise from finite irreducible systems closed under sector operations in the endomorphisms of Cuntz algebras. We describe his construction below.

Let G be a finite abelian group of order n with a non-degenerate symmetric pairing $\langle\ ,\ \rangle : G\times G \longrightarrow \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z|=1\}$. As an infinite factor $\mathcal{M}$, we adopt the Cuntz algebra $\mathcal{O}_{2n}$, which is the simple C^*-algebra generated by $\{S_g,\ T_h \mid g,h \in G\}$ with relations $S_g^*S_h = T_g^*T_h = \delta_{gh}1$, $S_g^*T_h = T_g^*S_h = 0$ $(g,h \in G)$ and $\sum_{g\in G} S_gS_g^* + \sum_{g\in G} T_gT_g^* = 1$. We consider two functions $a : G \longrightarrow \mathbb{T}$, $b : G \longrightarrow \mathbb{C}$ and an element $c \in \mathbb{T}$ satisfying the following conditions (A1) – (A7).

(A1) $a(0)=1,\ a(g)=a(-g),\ a(g+h)\langle g,h\rangle=a(g)a(h) \qquad (g,h\in G)$

(A2) $a(g)b(-g)=\overline{b(g)} \qquad (g,h\in G)$

(A3) $\frac{c\sqrt{n}}{d}+\sum_{g\in G}b(g)=0$

(A4) $\sum_{g\in G}b(g+h)\overline{b(g)}=\delta_{h,0}-\frac{1}{d} \qquad (h\in G)$

(A5) $c^3\hat{a}(0)=1$

(A6) $\hat{b}(g)=c\overline{b(g)} \quad (g\in G)$

(A7) $\sum_{g\in G}b(g+h)b(g+k)\overline{b(g)}=\overline{\langle h,k\rangle}b(h)b(k)-\frac{c}{d\sqrt{n}} \quad (h,k\in G)$

Here, $d=\frac{n+\sqrt{n^2+4n}}{2}$, that is a solution of the equation $d^2=nd+n$, and $\hat{a},\hat{b}$ are Fourier transformations given by the formula

$$\hat{f}(g)=\frac{1}{\sqrt{n}}\sum_{h\in G}\overline{\langle g,h\rangle}f(h) \qquad (g\in G)$$

for $f=a,b$. Then $*$-preserving endomorphisms α_g $(g\in G)$ and ρ are defined by

$$\alpha_g(S_h)=S_{g+h}, \qquad \alpha_g(T_h)=\langle g,h\rangle T_h \qquad (h\in G),$$
$$\rho(S_g)=\Big[\frac{1}{d}\sum_{h\in G}\langle g,h\rangle S_h+\frac{1}{\sqrt{d}}\sum_{h\in G}a(h)T_{h-g}T_{-h}\Big]U(g)^*,$$
$$\rho(T_g)=\frac{c}{\sqrt{nd}}\sum_{h,k\in G}\langle k,g\rangle\overline{\langle h,k\rangle}S_hT_k^*+\frac{\overline{a(g)c}}{\sqrt{n}}\sum_{h,k\in G}\langle h,g\rangle\langle h,k\rangle T_hS_kS_k^*$$
$$+\sum_{h,k\in G}a(h)b(g+h)\langle k,g\rangle T_{h+k}T_{-h}T_k^*,$$

where

$$U(g)=\sum_{h\in G}\langle g,h\rangle S_hS_h^*+\sum_{h\in G}T_{h-g}T_h^*,$$

which defines a unitary representation of G. It is easy to see that $\alpha_0=\mathrm{id},\ \alpha_g\cdot\alpha_h=\alpha_{g+h}$, $\alpha_g\cdot\rho=\rho$, $(\rho\cdot\alpha_g)(x)=U(g)\rho(x)U(g)^*$, $\rho^2(x)=\sum_{h\in G}S_h\alpha_h(x)S_h^*+\sum_{h\in G}T_h\rho(x)T_h^*$ for all $g,h\in G,\ x\in\mathcal{O}_{2n}$, and moreover, $\overline{\alpha_g}=\alpha_{-g},\ \bar{\rho}=\rho,\ R_{\alpha_g}=\overline{R}_{\alpha_g}=1,\ R_\rho=\overline{R}_\rho=S_0$, and $d(\alpha_g)=1,\ d(\rho)=d$ for all $g\in G$. Thus the subset $\Delta_{G,a,b,c}:=\{\alpha_g\mid g\in G\}\cup\{\rho\}\subset\mathrm{End}_0(\mathcal{O}_{2n})$ is a finite irreducible system closed under sector operations.

Let $\mathcal{M}$ be the weak closure of $\mathcal{O}_{2n}$ in the GNS representations considered in [3]. Then ρ can be extended to an endomorphism on $\mathcal{M}$, and a subfactor $\mathcal{N}\subset\mathcal{M}$ is obtained from the von Neumann algebra generated by $\rho(\mathcal{M})$ and $\{U(g)\}_{g\in G}$. This subfactor is called a generalized E_6-subfactor since in the case where $G=\mathbb{Z}/2\mathbb{Z}=\{0,1\}$ the subfactor $\mathcal{N}\subset\mathcal{M}$ arising from $\Delta_{G,a,b,c}$ is an E_6-subfactor. In addition to this example, Izumi gives several solutions of (A1) – (A7) in the case where G is a cyclic group of order $n\leq 7$ and the direct product $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ [6, 7].

§2. The definition of Turaev-Viro-Ocneanu invariant for 3-manifolds

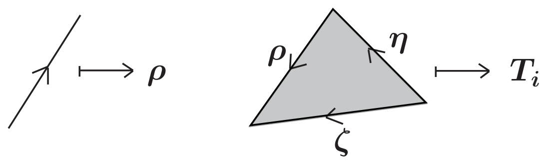
In this section, we review the definition of Turaev-Viro-Ocneanu invariant of 3-manifolds in the setting of sectors [1, 10].

Let Δ be a finite irreducible system of $\mathrm{End}_0(\mathcal{M})$ closed under sector operations. For $\rho, \eta, \zeta \in \Delta$, we set $\mathcal{H}_{\rho\eta}^{\zeta} = \mathrm{Hom}(\zeta, \rho\eta)$, and fix an orthonormal basis $\mathcal{B}_{\rho\eta}^{\zeta} = \{T_i\}$ of $\mathcal{H}_{\rho\eta}^{\zeta}$ satisfying the condition $(*)$ in the previous section.

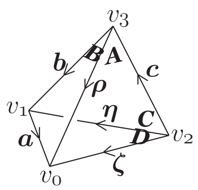
Let $\mathcal{K}$ be a simplicial complex, and suppose that each 1-simplex in $\mathcal{K}$ is oriented so that a cycle does not appear in any 2-simplex. A map

$$\varphi : (\{\text{the 1-simplices in } \mathcal{K}\},\ \{\text{the 2-simplices in } \mathcal{K}\}) \longrightarrow \Big(\Delta,\ \bigcup_{\rho,\eta,\zeta\in\Delta} \mathcal{B}_{\rho\eta}^{\zeta}\Big)$$

is called a color of $\mathcal{K}$ if $\varphi(|v_0v_1v_2|)$ belongs to $\mathcal{B}_{\rho\eta}^{\zeta}$ for a 2-simplex $|v_0v_1v_2| \in \mathcal{K}$, where $\varphi(\langle v_0, v_1\rangle) = \rho$, $\varphi(\langle v_1, v_2\rangle) = \eta$, $\varphi(\langle v_0, v_2\rangle) = \zeta$, and $\langle v_i, v_j\rangle$ denotes the oriented 1-simplex.



Let M be a compact oriented 3-manifold whose boundary is triangulated by a simplicial complex $\mathcal{K}$, supposed that each edge in $\mathcal{K}$ is oriented so that a cycle does not appear in every triangle. Let $\mathcal{T}$ be a triangulation of M satisfying with the same condition as $\mathcal{K}$, and that $\mathcal{T}$ coincides with $\mathcal{K}$ on the boundary ∂M. For a colored tetrahedron $\sigma =$



in $\mathcal{T}$, we define a complex number called a quantum $6j$-symbol by

$$\frac{1}{\sqrt{d(\rho)d(\eta)}} A^* B^* a(C) D \quad \in \quad \mathrm{Hom}(\zeta, \zeta) \cong \mathbb{C}.$$

We denote the above complex number or its complex conjugate by $W(\sigma; \varphi)$ according to compatibility of orientations for M and σ. Here, the orientation for σ is given by the order $v_0 < v_1 < v_2 < v_3$.

For a color ψ of $\mathcal{K}$, we set

$$Z^{\Delta}(M; \mathcal{T}, \psi) = \lambda^{-\sharp\mathcal{T}^{(0)} + \frac{\sharp\mathcal{K}^{(0)}}{2}} \sqrt{d(\psi)} \sum_{\substack{\varphi\,:\,\text{colors of } \mathcal{T} \\ \varphi|_{\mathcal{K}} = \psi}} d(\varphi|_{\mathcal{T}-\mathcal{K}}) \prod_{\sigma\,:\,\text{tetrahedra of } \mathcal{T}} W(\sigma; \varphi),$$

where $\lambda = \sum_{\rho\in\Delta} d(\rho)^2$, which is called the global index of Δ, and

$$d(\psi) = \prod_{e\,:\,\text{edges of } \mathcal{K}} d(\psi(e)), \qquad d(\varphi|_{\mathcal{T}-\mathcal{K}}) = \prod_{e\,:\,\text{edges of } \mathcal{T}-\mathcal{K}} d(\varphi(e)).$$

By the Frobenius reciprocity of sectors established by Izumi [4], it can be shown that the complex number $Z^{\Delta}(M; \mathcal{T}, \psi)$ does not depend on the choice of orientations for edges in $\mathcal{T}$. However, the pentagon identity does not hold in general [1, Chapter 12]. For Δ which pentagon identities hold for all $a, b, c, e, f, j, k, l \in \Delta$ and A, B, C, E, F, G, the complex number $Z^{\Delta}(M; \mathcal{T}, \psi)$ becomes a topological invariant of M with a fixed triangulation $\mathcal{K}$ of ∂M and its color ψ. In this case, we write $Z^{\Delta}(M;\psi)$ instead of $Z^{\Delta}(M;\mathcal{T},\psi)$, and refer to it as the Turaev-Viro-Ocneanu invariant of (M,ψ). In the case where $\partial M = \varnothing$, we denote the Turaev-Viro-Ocneanu invariant $Z^{\Delta}(M;\psi)$ by $Z^{\Delta}(M)$ since there is no color of the boundary.

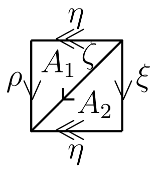
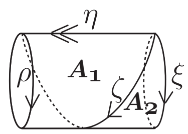
Since any finite irreducible system $\Delta_{G,a,b,c}$ introduced in the previous section satisfies the pentagon identities thanks to the conditions (A1) – (A7), we have a Turaev-Viro-Ocneanu invariant from $\Delta_{G,a,b,c}$.

§3. Tube algebras

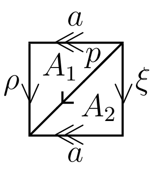
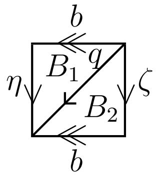
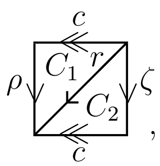
The concept of the tube algebra, which plays a crucial role in the Turaev-Viro-Ocneanu TQFT, was first introduced by Ocneanu [10]. Here, we review the definition of Ocneanu's tube algebra (see also [8] for precisely definition).

Let $\mathcal{M}$ be an infinite factor, and Δ a finite irreducible system of $\mathrm{End}_0(\mathcal{M})$ satisfying pentagon identities. We set

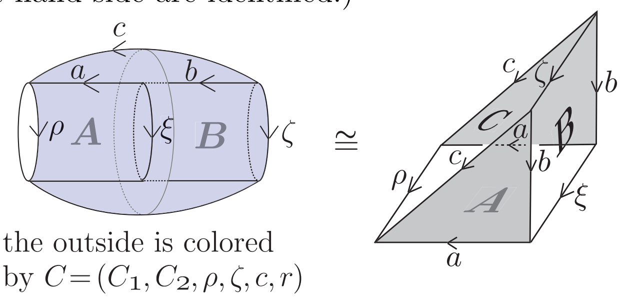
$$\mathrm{Tube}\,\Delta = \bigoplus_{\rho,\xi,\zeta,\eta\in\Delta} \mathcal{H}^{\zeta}_{\rho\eta}\otimes\mathcal{H}^{\zeta}_{\eta\xi}.$$

For $A_1 \in \mathcal{H}^{\zeta}_{\rho\eta}$, $A_2 \in \mathcal{H}^{\zeta}_{\eta\xi}$ we represent the element $A_1\otimes A_2 \in \mathcal{H}^{\zeta}_{\rho\eta}\otimes\mathcal{H}^{\zeta}_{\eta\xi}$ by the figure . It can be regarded as a tube .

Then, $\mathrm{Tube}\,\Delta$ is an algebra over $\mathbb{C}$ whose product $\star$ is given by

 $\star$  $= \dfrac{\delta_{\xi,\eta}\lambda}{\sqrt{d(\rho)d(\zeta)d(\xi)}} \displaystyle\sum_{c,r\in\Delta}\sum_{\substack{C_1\in\mathcal{B}^r_{\rho c}\\ C_2\in\mathcal{B}^r_{c\zeta}}} Z^{\Delta}(\mathbb{D}^2\times\mathbb{S}^1,\psi)$ ,

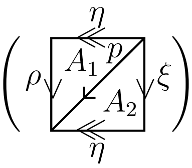
where $\delta_{\xi,\eta}$ is Kronecker's delta, and ψ is a color of the boundary of the triangulation of the solid torus $\mathbb{D}^2\times\mathbb{S}^1$ illustrated as in the figure below. (Here, the two shaded triangles in the right-hand side are identified.)



the outside is colored by $C=(C_1,C_2,\rho,\zeta,c,r)$

Moreover, $\mathrm{Tube}\,\Delta$ has a structure of C^*-algebra whose $*$-operation is defined by inversing the tube inside out. We call this C^*-algebra the tube algebra in the Turaev-Viro-Ocneanu TQFT Z^{Δ}. The algebra $\mathrm{Tube}\,\Delta$ is semisimple since a finite-dimensional C^*-algebra over $\mathbb{C}$ is semisimple.

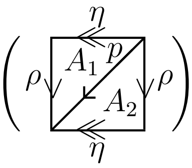
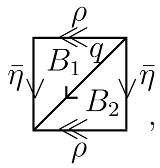
Izumi [4] introduced the tube algebra in the setting of sectors, and showed that there is a faithful positive linear functional on $\mathrm{Tube}\,\Delta$. The functional, denoted by φ_Δ, is given by

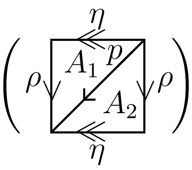
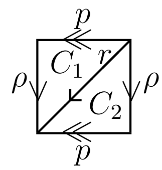
$$\varphi_\Delta\Bigg($$

$$\Bigg) = d(\xi)^2 \delta_{\rho,\xi}\delta_{\eta,\mathrm{id}} A_2 A_1^*.$$

We note that the right-hand side, actually, is a complex number since $A_2A_1^* \in \mathrm{Hom}(\rho,\rho) \cong \mathbb{C}$.

Let $\{z_i\}_{i=0}^m$ be the set of the primitive idempotents of the center $\mathcal{Z}(\mathrm{Tube}\,\Delta)$ of $\mathrm{Tube}\,\Delta$. Since $\mathcal{Z}(\mathrm{Tube}\,\Delta)$ is a commutative semisimple algebra, $\{z_i\}_{i=0}^m$ is a basis of $\mathcal{Z}(\mathrm{Tube}\,\Delta)$. It is easily proved that $\varphi(z_i)$ is a positive real number for each i. So, we set $d(i) = \sqrt{\lambda\varphi(z_i)}$, where $\lambda = \sum_{\rho\in\Delta} d(\rho)^2$.

Let $\mathrm{SL}(2,\mathbb{Z})$ be the group consisting of 2×2-matrices of integer coefficients with determinant 1. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on the center $\mathcal{Z}(\mathrm{Tube}\,\Delta)$ as follows [5].

$$S'_\Delta\Bigg($$

$$\Bigg) = d(\rho)\sum_{q\in\Delta}\sum_{\substack{B_1\in\mathcal{B}^q_{\bar\eta\rho}\\ B_2\in\mathcal{B}^q_{\rho\bar\eta}}} B_2^* X B_1$$

,

$${T'_\Delta}^{-1}\Bigg($$

$$\Bigg) = \sum_{r\in\Delta}\sum_{\substack{C_1\in\mathcal{B}^r_{\rho p}\\ C_2\in\mathcal{B}^r_{p\rho}}} C_2^* Y C_1$$


for $S' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $X = R_\eta^*\bar\eta(A_2A_1^*\rho(\bar R_\eta)) \in \mathrm{Hom}(\bar\eta\rho,\rho\bar\eta)$, $Y = A_1^*\rho(A_2) \in \mathrm{Hom}(\rho p, p\rho)$. We remark that $B_2^*XB_1 \in \mathrm{Hom}(q,q) = \mathbb{C}$, $C_2^*YC_1 \in \mathrm{Hom}(r,r) = \mathbb{C}$. With respect to the basis $\{\frac{\sqrt{\lambda}}{d(i)} z_i\}_{i=0}^m$ of $\mathcal{Z}(\mathrm{Tube}\,\Delta)$, we may write

$$S'_\Delta\Big(\frac{\sqrt{\lambda}}{d(i)} z_i\Big) = \sum_{j=0}^m S_{ji}\frac{\sqrt{\lambda}}{d(j)} z_j \qquad (S_{ji}\in\mathbb{C}), \tag{3.1}$$

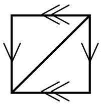
$$T'_\Delta\Big(\frac{\sqrt{\lambda}}{d(i)} z_i\Big) = t_i\frac{\sqrt{\lambda}}{d(i)} z_i \qquad (t_i\in\mathbb{C}), \tag{3.2}$$

since the linear map T'_Δ is represented by a diagonal matrix [5].

§4. Formulas of Turaev-Viro-Ocneanu invariants for lens spaces

In this section, we explain a method to compute the Turaev-Viro-Ocneanu invariant derived from subfactors. Our method is based on the Dehn surgery formula in $(2+1)$-dimensional topological quantum field theory with Verlinde basis [8]. In what follows, we only consider finite irreducible systems Δ satisfying pentagon identities.

The Turaev-Viro-Ocneanu TQFT Z^Δ derived from Δ assigns each (triangulated) oriented closed surface Σ to a finite-dimensional vector space $Z^\Delta(\Sigma)$. In the case where Σ is the torus $\mathbb{T}^2 = \mathbb{S}^1\times\mathbb{S}^1$, the vector space $Z^\Delta(\mathbb{T}^2)$ is defined as follows. We regard

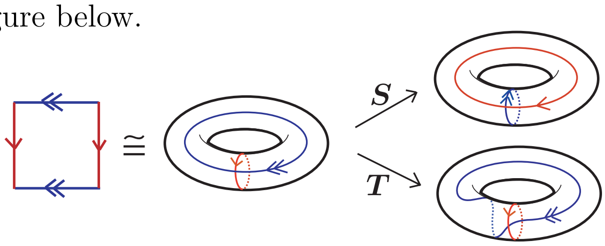
$\mathbb{T}^2$ as a topological space obtained by identifying with opposite sides of a square in a usual way, and consider the singular triangulation $\mathcal{K}$ =  of $\mathbb{T}^2$.

Let $V^{\Delta}(\mathbb{T}^2)$ denote the vector space freely spanned by the colors of $\mathcal{K}$ over $\mathbb{C}$. The vector space $V^{\Delta}(\mathbb{T}^2)$ is identified with the subspace $\bigoplus_{\rho,a,p\in\Delta} \mathcal{H}^p_{\rho a} \otimes \mathcal{H}^p_{a\rho} \subset \mathrm{Tube}\,\Delta$.

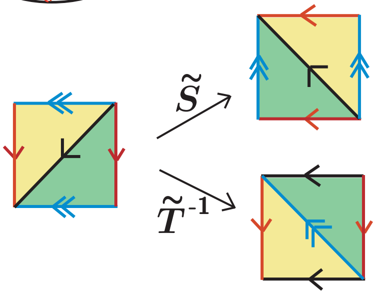
Let $\Phi : V^{\Delta}(\mathbb{T}^2) \longrightarrow V^{\Delta}(\mathbb{T}^2)$ denote the linear map defined by

$$\Phi(\psi_0) = \sum_{\psi_1\,:\,\text{colors}} Z^{\Delta}(\mathbb{T}^2 \times [0,1]; \psi_0 \sqcup \psi_1)\, \psi_1$$

for all colors ψ_0 of $\mathcal{K}$, where $Z^{\Delta}(\mathbb{T}^2\times[0,1];\psi_0\sqcup\psi_1)$ is the Turaev-Viro-Ocneanu invariant of $\mathbb{T}^2 \times [0,1]$ whose boundary is colored by ψ_t on $\mathbb{T}^2 \times \{t\}$ for $t = 0, 1$. Then, we set $Z^{\Delta}(\mathbb{T}^2) = \mathrm{Im}\,\Phi \subset V^{\Delta}(\mathbb{T}^2)$. By the method of construction of Turaev-Viro-Ocneanu invariants for 3-manifolds with boundaries, we see that the mapping class group of $\mathbb{T}^2$, which is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, acts on the vector space $Z^{\Delta}(\mathbb{T}^2)$. This action is given by the following. Let S, T be the orientation preserving homeomorphisms on $\mathbb{T}^2$ depicted as in the figure below.



Then, the lifts of S, T^{-1} with respect to the universal covering $\mathbb{R}^2 \longrightarrow \mathbb{T}^2$ are given by $\tilde{S}(x,y) = (y,-x)$, $\tilde{T}^{-1}(x,y) = (x,-x+y)$, respectively. We observe that $\tilde{S}$, $\tilde{T}^{-1}$ are simplicial maps from $\mathcal{K}$ to the singular triangulations $\mathcal{L}_S, \mathcal{L}_{T^{-1}}$ depicted as in the right figure, respectively.

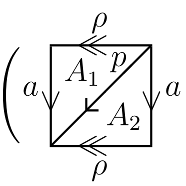
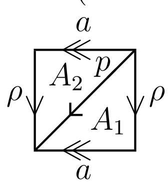


For $f \in \{S, T^{-1}\}$, a linear map $f_\sharp : V^{\Delta}(\mathbb{T}^2) \longrightarrow V^{\Delta}(\mathbb{T}^2)$ is defined by

$$f_\sharp(\psi_0) = \sum_{\psi_1\,:\,\text{colors}} Z^{\Delta}(\mathbb{T}^2 \times [0,1]; f\psi_0 \sqcup \psi_1)\, \psi_1$$

for all colors ψ_0 of $\mathcal{K}$, where $f\psi_0$ is the color of $\mathcal{L}_f$ determined by ψ_0 and f, and $Z^{\Delta}(\mathbb{T}^2 \times [0,1]; f\psi_0 \sqcup \psi_1)$ is the Turaev-Viro-Ocneanu invariant of $\mathbb{T}^2 \times [0,1]$ whose boundary is colored by $f\psi_0$ and ψ_1 on $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$, respectively. This linear map $f_\sharp$ induces a linear isomorphism $Z^{\Delta}(f) : Z^{\Delta}(\mathbb{T}^2) \longrightarrow Z^{\Delta}(\mathbb{T}^2)$, and the map $f \longmapsto Z^{\Delta}(f)$ gives a representation of $\mathrm{SL}(2,\mathbb{Z})$ on the space $Z^{\Delta}(\mathbb{T}^2)$.

Let us consider a conjugate-linear map $\phi : V^{\Delta}(\mathbb{T}^2) \longrightarrow \mathcal{Z}(\mathrm{Tube}\,\Delta)$ defined by

$\phi\Bigg($  $\Bigg) = \lambda^{-\frac{1}{2}} \sqrt{\dfrac{d(a)}{d(\rho)d(p)}}$ 

for all $a, \rho, p \in \Delta$, $A_1 \in \mathcal{B}^p_{\rho a}$, $A_2 \in \mathcal{B}^p_{a\rho}$. This map ϕ induces a conjugate-linear isomorphism $Z^\Delta(\mathbb{T}^2) \longrightarrow \mathcal{Z}(\mathrm{Tube}\,\Delta)$ [11]. We set

$$v_i = \frac{\lambda}{d(i)} Z^\Delta(S)(\phi^{-1}(z_i)) \qquad (i = 0, 1, \cdots, m).$$

Combining results in [8] and [11], then we have :

Theorem(**Kawahigashi-Sato-W.**[8], **Sato-W.**[11]). *The basis $\{v_i\}_{i=0}^m$ of $Z^\Delta(\mathbb{T}^2)$ is a Verlinde basis associated with the Turaev-Viro-Ocneanu TQFT Z^Δ, and*

$$(Z^\Delta(T))(v_j) = \overline{t_j}\, v_i, \qquad (Z^\Delta(S))(v_j) = \sum_{i=0}^m \overline{S_{ji}}\, v_i \qquad (j = 0, 1, \cdots, m),$$

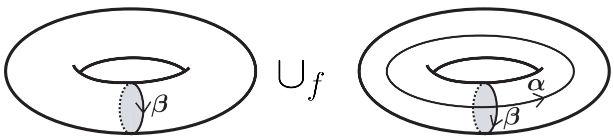
where t_j and S_{ji} are complex numbers defined in (3.1) and (3.2).

We can choose as v_0 in a Verlinde basis $\{v_i\}_{i=0}^m$ the element

$$1 = \sum_{\psi\,:\,\text{colors}} Z^\Delta(\mathbb{D}^2 \times \mathbb{S}^1; \psi)\, \psi \;\in\; Z^\Delta(\mathbb{T}^2),$$

which is the identity element of the fusion algebra associated to Z^Δ.

For a pair (p, q) of coprime integers, the lens space $L(p, q)$ is obtained from two solid tori $\mathbb{D}^2 \times \mathbb{S}^1$ by gluing their boundaries along a homeomorphism $f : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such as $H_1(f)([\beta]) = p[\alpha] + q[\beta]$, where $H_1(f)$ is the induced homomorphism from the 1-dimensional homology group $H_1(\mathbb{T}^2)$ to itself, and (β, α) is a standard meridian-longitude system.

$L(p, q) \;=$ 

Then, we have $Z^\Delta(L(p, q)) = \langle f_*(1), 1\rangle$, where the bracket is a bilinear from on $Z^\Delta(\mathbb{T}^2)$ defined by $\langle v_i, v_j\rangle = \delta_{ij}$. If $q = 1$, then we have $\overline{Z^\Delta(L(p, 1))} = \sum_{i=0}^m t_i^p S_{i0}^2$ by taking $f = S \circ T^p \circ S$, and if $p \equiv -1 \pmod q$, then we have $\overline{Z^\Delta(L(p, q))} = \sum_{i,j=0}^m t_i^{\frac{p+1}{q}} t_j^q S_{i0} S_{j0} S_{ij}$ by taking $f = S \circ T^{\frac{p+1}{q}} \circ S \circ T^q \circ S$. Right-hand sides of these formulas are rewritten by using φ_Δ as follows.

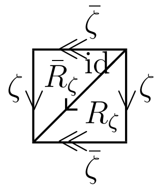
Proposition. *We set $\boldsymbol{t}' = \sum_{i=0}^m t_i z_i \in \mathcal{Z}(\mathit{Tube}\,\Delta)$. Then, we have*

(1) $\overline{Z^\Delta(L(p, 1))} = \dfrac{1}{\lambda}\varphi_\Delta(\boldsymbol{t}'^p)$.

(2) If $p \equiv -1$ (mod q), then $\overline{Z^\Delta(L(p, q))} = \dfrac{1}{\lambda}\varphi_\Delta(\boldsymbol{t}'^{\frac{p+1}{q}} S'_\Delta(\boldsymbol{t}'^q))$.

In particular, we have formulas for $\overline{Z^\Delta(L(p, 2))}$ and $\overline{Z^\Delta(L(7, 4))}$.

The formula of Part (1) in Proposition has already appeared in [6].

Applying the above formulas in case of $\Delta = \Delta_{G,a,b,c}$ defined in Section 1, and substituting $\boldsymbol{t}' = \sum_{\zeta\in\Delta} d(\zeta)$ 

, we have formulas to compute $Z^{\Delta_{G,a,b,c}}(L(p,q))$ ($q = 1,2$) and $Z^{\Delta_{G,a,b,c}}(L(7,4))$ in terms of initial data of $\Delta_{G,a,b,c}$. For example, $Z^{\Delta_{G,a,b,c}}(L(7,4))$ can be computed by

$$\begin{aligned}
Z^{\Delta_{G,a,b,c}}(L(7,4)) = \frac{1}{\lambda}\{ & n_7 + \frac{c^2}{d}\sum_{g,k\in G} a(k)^2 a(g)^4 a(g+k) \\
& + \sum_{g,h\in G} a(g)\overline{b(g+h)\langle h, g-h\rangle}\sum_{k\in G}\overline{b(k-2g+h)\langle k-h,k\rangle} \\
& + c^2\sum_{k,l\in G}\overline{a(k)b(l-k)}\sum_{g\in G} a(g)^5\overline{b(2k-g+l)\langle l+k-g,l\rangle} \\
& + d\sum_{g,k\in G} a(g)^6\overline{a(k-g)}\sum_{l\in G}\overline{b(l-2k)b(k-g+l)}\langle l, g+k-l\rangle \\
& \qquad \times \sum_{h\in G} b(h)b(3g+h-l)\}.
\end{aligned}$$

Here, $d = \frac{n+\sqrt{n^2+4n}}{2}$, $\lambda = n + d^2$, $n_7 = \sharp\{g \in G \mid 7g = 0\}$.

Recently, Izumi [7] gave new several solutions for the system of equations (A1) — (A7). Let us denote new finite irreducible systems from these solutions by $\Delta_{5,\varepsilon_1,\varepsilon_2}$ ($\varepsilon_1, \varepsilon_2 = \pm 1$), $\Delta_{6,\varepsilon}$ ($\varepsilon = 0,1$), Δ_7. They are given by the following.

- $\Delta_{5,\varepsilon_1,\varepsilon_2} = \Delta_{G,a,b,c}$ ($\varepsilon_1, \varepsilon_2 \in \{-1,1\}$) :
  $G = \mathbb{Z}/5\mathbb{Z}$, $\zeta = \exp(\frac{2\pi\sqrt{-1}}{5})$, $\langle g,h\rangle = \zeta^{2gh}$, $a(g) = \zeta^{-g^2}$,
  $b(0) = -\frac{1}{d}$, $b(1) = \frac{\zeta^3\eta_1}{\sqrt{5}}$, $b(2) = \frac{\zeta^2\eta_2}{\sqrt{5}}$, $c = \frac{-1-\varepsilon_1\varepsilon_2\sqrt{-3}}{2}$,
  $\eta_j = \frac{-1-\varepsilon_1\varepsilon_2\sqrt{15-6\sqrt{5}}+\varepsilon_j i\sqrt{6\sqrt{5}+(-1)^j\varepsilon_1\varepsilon_2 2\sqrt{15-6\sqrt{5}}}}{4} \qquad (j = 1,2)$
- $\Delta_{6,\varepsilon} = \Delta_{G,a,b,c}$ ($\varepsilon \in \{0,1\}$) :
  $G = \mathbb{Z}/6\mathbb{Z}$, $\zeta = \exp(\frac{2\pi\sqrt{-1}}{24})$, $\langle g,h\rangle = \zeta^{4gh}$, $a(g) = (-1)^{g\varepsilon}\zeta^{-2g^2}$,
  $b(0) = -\frac{1}{d}$, $b(1) = \frac{(-\sqrt{-1})^\varepsilon\zeta\eta_1}{\sqrt{6}}$, $b(2) = \frac{\zeta^4\eta_2}{\sqrt{6}}$, $b(3) = \frac{(-\sqrt{-1})^\varepsilon\zeta^{-3}}{\sqrt{6}}$, $c = (-\sqrt{-1})^\varepsilon\zeta$,
  $\eta_1 = \frac{2-(-1)^\varepsilon\sqrt{3}-\sqrt{15}}{4} + i\frac{\sqrt{(-1)^\varepsilon 2\sqrt{3}+2\sqrt{15}-3-(-1)^\varepsilon 3\sqrt{5}}}{2\sqrt{2}}$,
  $\eta_2 = \frac{(-1)^\varepsilon 3+\sqrt{3}+\sqrt{5}-(-1)^\varepsilon\sqrt{15}}{4\sqrt{2}} - i\frac{\sqrt{\sqrt{15}+(-1)^\varepsilon\sqrt{3}}}{2\sqrt{2}}$
- $\Delta_7 = \Delta_{G,a,b,c}$:
  $G = \mathbb{Z}/7\mathbb{Z}$, $\zeta = \exp(\frac{2\pi\sqrt{-1}}{7})$, $\langle g,h\rangle = \zeta^{gh}$, $a(g) = \zeta^{3g^2}$,

$$b(0)=-\frac{1}{d},\ b(1)=\frac{\zeta^2\eta_1}{\sqrt{7}},\ b(2)=\frac{\zeta\eta_2}{\sqrt{7}},\ b(3)=\frac{\zeta^4\eta_3}{\sqrt{7}},\ c=-\sqrt{-1},$$
$$\eta_1=\frac{-\sqrt{7}(\zeta+\zeta^{-1})^2-\sqrt{11}(\zeta-\zeta^{-1})^2+(\zeta^2-\zeta^{-2})\sqrt{2\sqrt{77}-14}}{4(\zeta-\zeta^{-1})(\zeta^4-\zeta^{-4})},$$
$$\eta_2=\frac{-\sqrt{7}(\zeta^4+\zeta^{-4})^2-\sqrt{11}(\zeta^4-\zeta^{-4})^2+(\zeta-\zeta)\sqrt{2\sqrt{77}-14}}{4(\zeta^2-\zeta^{-2})(\zeta^4-\zeta^{-4})},$$
$$\eta_3=\frac{-\sqrt{7}(\zeta^2+\zeta^{-2})^2-\sqrt{11}(\zeta^2-\zeta^{-2})^2+(\zeta^4-\zeta^{-4})\sqrt{2\sqrt{77}-14}}{4(\zeta-\zeta^{-1})(\zeta^2-\zeta^{-2})}$$

By partially using the Maple software Release 5, we computed Turaev-Viro-Ocneanu invariants of lens spaces $L(p,q)$ for $p\leq 7$ in each case of new finite irreducible systems defined above. The following table is one of the results of our computations.

	$\Delta_{5,\varepsilon_1,\varepsilon_2}$	$\Delta_{6,\varepsilon}$	Δ_7
$L(3,1)$	$\frac{3+\sqrt{-1}\varepsilon_1\varepsilon_2\sqrt{15}}{30}$	$\frac{5-\sqrt{-1}\sqrt{5}}{20}$	$\frac{11+\sqrt{77}}{154}$
$L(5,1)$	$\frac{2}{3}$	$\frac{1-(-1)^{\varepsilon}\sqrt{3}}{12}$	$\frac{11+\sqrt{77}}{154}$
$L(5,2)$	$\frac{1}{3}$	$\frac{1+(-1)^{\varepsilon}\sqrt{3}}{12}$	$\frac{11+\sqrt{77}}{154}$
$L(7,q),\ q=1,2$	$\frac{3+\sqrt{5}}{30}$	$\frac{5-\sqrt{15}}{60}$	$\frac{11-\sqrt{-1}\sqrt{11}}{22}$

We remark that $Z^{\Delta}(L(p,q))=\overline{Z^{\Delta}(L(p,p-q))}$ since $L(p,p-q)$ is homeomorphic to $L(p,q)$ with opposite orientation, and $Z^{\Delta}(L(7,4))=Z^{\Delta}(L(7,2))$ since $L(7,4)$ is orientation preserving homeomorphic to $L(7,2)$. As a result, we see that the Turaev-Viro-Ocneanu invariant derived from the generalized E_6-subfactor with group symmetry $G=\mathbb{Z}/7\mathbb{Z}$ dose not distinguish the lens spaces $L(7,1)$ and $L(7,2)$ ($\cong L(7,4)$).

References

[1] D.E.Evans and Y.Kawahigashi, *Quantum symmetries on operator algebras*, Oxford University Press (1998)

[2] M. Izumi, *Applications of fusion rules to classification of subfactors*, Publ. RIMS **27** (1991) p.953–994.

[3] M. Izumi, *Subalgebras of finite C^*-algebras with finite Watatani indices, I: Cuntz algebras*, Commun. Math.Phys. **155** (1993) p.157–182.

[4] M. Izumi, *Subalgebras of finite C^*-algebras with finite Watatani indices, II: Cuntz-Krieger algebras*, Duke Math. J. **91** (1998) 409–461.

[5] M. Izumi, *The structures of sectors associated with the Longo-Rehren inclusions I. General theory*, Commun. Math.Phys. **213** (2000) p.127—179.

[6] M. Izumi, *The structures of sectors associated with the Longo-Rehren inclusions II. Examples*, Review in Math.Phys. **13** (2001) p.603—674.

[7] M. Izumi, unpublished manuscript.

[8] Y. Kawahigashi, N. Sato and M. Wakui, $(2+1)$*-dimensional topological quantum field theory from subfactors and Dehn surgery formula for* 3*-manifold invariants*, Adv. in Math. **195** (2005) p.165–204.

[9] R. Longo, *Index of subfactors and statistics of quantum fields II. Correspondences, braid group statistics and Jones polynomial*, Commun. in Math. Phys. **130** (1990) p.285–309.

[10] A. Ocneanu, *Chirality for operator algebras*, in "Subfactors", ed. by H. Araki, et al., World Scientific, 1994, p.39—63.

[11] N. Sato and M. Wakui, *Computations of Turaev-Viro-Ocneanu invariants of* 3*-manifolds from subfactors*, JKTR **12** (2003) p.543–574.

[12] V.G.Turaev and O.Ya.Viro, *State sum invariants of 3-manifolds and quantum* $6j$*-symbols*, Topology **31** (1992) p.865—902.

Department of Mathematics, Faculty of Engineering Science
Kansai University
3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680, Japan