Indecomposability of weak Hopf algebras

Michihisa Wakui

This paper is dedicated to Professor Toshitake Kohno on his 65th birthday.

Abstract. In this paper we discuss on the direct sum construction of weak bialgebras and indecomposability of them as weak bialgebras. Many fundamental properties and examples of the direct sum construction of weak bialgebras and indecomposable weak bialgebras are investigated. For example, any finite-dimensional weak bialgebra can be uniquely decomposed into finitely many indecomposable weak bialgebras up to isomorphism. A finite-dimensional bialgebra is always indecomposable as a weak bialgebra. The direct sum construction and indecomposability can be characterized in language of category theory, and from this point of view a generalization of the direct sum construction is introduced.

1. Introduction

Weak Hopf algebras are generalizations of Hopf algebras, and the general theory of them is established by Böhm, Nill and Szlachányi [5] in 1999. They are also called quantum groupoids in some literatures, and it is known by Schauenburg [22] that they can be regarded as certain $\mathbb{R}$-bialgebras introduced by Takeuchi [24] in 1977. Face algebras which are introduced by Hayashi [8] give a wide class in weak Hopf algebras and are important from a historical point of view.

In the present paper we focus on the direct sum construction of weak bialgebras and indecomposability of them as weak bialgebras. In general, for weak bialgebras $A$ and $B$, the direct sum $A \oplus B$ is also a weak bialgebra with respect to the direct product as an algebra and the direct sum as a coalgebra. This fact is remarkable since the same result does not hold for ordinary bialgebras. The direct sum construction for weak bialgebras leads to decomposing weak bialgebras into indecomposable ones. In fact, any finite-dimensional weak bialgebra can be uniquely decomposed into finitely many indecomposable ones up to isomorphism. This shows that indecomposable weak bialgebras are fundamental and important in study of weak bialgebras.

2010 Mathematics Subject Classification. Primary 16T05. 18D10, 18D25.
Key words and phrases. weak bialgebra, weak Hopf algebra, indecomposable, direct sum, quantum double.
We investigate many fundamental properties and examples of the direct sum construction of weak bialgebras and indecomposable ones. For example, a finite-dimensional bialgebra is always indecomposable as a weak bialgebra. The Kaplansky type construction studied by Chebel and Makhlouf [7] can be regarded as a special direct sum construction. If a weak Hopf algebra is the direct sum of two weak Hopf algebras \( A \) and \( B \), then the quantum double of \( A \oplus B \) is naturally isomorphic to the direct sum of the quantum doubles of \( A \) and \( B \).

Indecomposability for weak bialgebras of finite dimension can be completely interpreted in language of category theory. More precisely, a finite-dimensional weak bialgebra \( H \) is indecomposable if and only if the module category \( H \mathcal{M} \) whose objects are finite-dimensional left \( H \)-modules is indecomposable. Along this principle we introduce a generalization of the direct sum construction involving weak bicomodule algebras, and give a categorical characterization of it.

This paper is organized as follows. In Section 2 we briefly describe definitions and basic properties of weak bialgebras which are used in the present paper. In Section 3 we introduce the direct sum construction of weak bialgebras and Hopf algebras. It is shown that any finite-dimensional weak bialgebra can be uniquely decomposed into finitely many indecomposable ones up to isomorphism, and that a finite-dimensional bialgebra is indecomposable as a weak bialgebra. Several examples including the Kaplansky type construction [7] and groupoid algebras are given. In Section 4 it is shown that any quasitriangular structure of the direct sum of two weak Hopf algebras \( A \) and \( B \) is a sum of quasitriangular structures of \( A \) and \( B \). It is also shown that the direct sum construction is preserved under the quantum double construction. In the final section we interpret the direct sum construction and indecomposability in language of category theory, and introduce a generalization of the direct sum construction. We demonstrate a few examples, and examine those indecomposability.

Throughout of the paper \( k \) denotes a field. All tensor products \( \otimes \) are taken over \( k \). For a weak bialgebra \( H \) over \( k \), we denote by \( H \mathcal{M} \) the \( k \)-linear category whose objects are left \( H \)-modules and morphisms are \( H \)-module maps between them. The \( k \)-linear categories of right \( H \)-comodules and finite-dimensional ones are denoted by \( H \mathcal{M}^H \) and \( \mathcal{M}^H \), respectively.

We use Sweedler’s notation such as \( \Delta(x) = x_{(1)} \otimes x_{(2)} \) for an element \( x \in H \) in a weak bialgebra \( H = (H, \Delta, \varepsilon) \). For a left \( H \)-comodule \( (M, \lambda) \) we use the notation such as \( \lambda(m) = m_{(-1)} \otimes m_{(0)} = m_{[-1]} \otimes m_{[0]} \) for an element \( m \in M \). For general facts on Hopf algebras and monoidal categories, we refer the reader to Abe’s book [1], Kassel’s book [11], MacLane’s book [12] and Montgomery’s book [14].

2. Definitions and basic properties related with weak bialgebras

Let \( H \) be a vector space over \( k \), and \( (H, \mu, \eta) \) and \( (H, \Delta, \varepsilon) \) are an algebra and a coalgebra over \( k \), respectively. The 5-tuple \( (H, \mu, \eta, \Delta, \varepsilon) \) is said to be a weak bialgebra over \( k \) if the following three conditions are satisfied.

(WH1) \( \Delta(xy) = \Delta(x)\Delta(y) \) for all \( x, y \in H \).
(WH2) \( \Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) \), where \( \Delta^{(2)} = (\Delta \otimes \id) \circ \Delta = (\id \otimes \Delta) \circ \Delta \) and \( 1 = \eta(1) \) is the identity element of the algebra \( (H, \mu, \eta) \).
(WH3) \(\varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \varepsilon(xy_{2}) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z)\) for all \(x, y, z \in H\).

Let \(S : H \rightarrow H\) be a \(k\)-linear map. The 6-tuple \((H, \mu, \eta, \Delta, \varepsilon, S)\) is said to be a weak Hopf algebra over \(k\) if the above three conditions and the following additional condition are satisfied.

(\text{WH4}) For all \(x \in H\)

(i) \(x_{(1)}S(x_{(2)}) = \varepsilon(1_{(1)}x)1_{(2)}\),
(ii) \(S(x_{(1)})x_{(2)} = 1_{(1)}\varepsilon(x1_{(2)})\),
(iii) \(S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x)\).

The above \(S\) is called the antipode of \((H, \mu, \eta, \Delta, \varepsilon)\) and \((H, \mu, \eta, \Delta, \varepsilon, S)\). We note that it is unique if it exists. We also note that a weak bialgebra is a bialgebra if and only if \(\Delta(1) = 1 \otimes 1\) if and only if \(\varepsilon\) is an algebra map.

For a weak bialgebra \(H = (H, \mu, \eta, \Delta, \varepsilon)\), by the right-hand sides of the conditions (\text{WH4})(i),(ii), the following two \(k\)-linear maps \(\varepsilon_t, \varepsilon_s : H \rightarrow H\) are defined:

\[
\begin{align*}
\varepsilon_t(x) &= \varepsilon(1_{(1)}x)1_{(2)}, \\
\varepsilon_s(x) &= 1_{(1)}\varepsilon(x1_{(2)}).
\end{align*}
\]

The maps \(\varepsilon_t\) and \(\varepsilon_s\) are called the target counital map and the source counital map, respectively.

\textbf{Example 2.1.} Let \(X\) be a non-empty set. By a small category with the object set \(X\) we mean a set \(M\) equipped with two maps \(s, t : M \rightarrow X\) and a binary operation \(\circ : M \times M \rightarrow M\), where \(M \times M = \{ (\alpha, \beta) \in M \times M \mid s(\alpha) = t(\beta) \}\), such that they become a category whose objects are \(X\) and morphisms are \(M\) and the composition is given by \(\circ\). We note that the above \(M\) can be viewed as a monoid in the category of spans over \(X\) (See [3, (2.6) and (5.4.3)]). For all \(x \in X\) we denote the identity morphism on \(x\) by \(1_x\). The small category \(M\) is called finite if the underlying set is finite, and \(M\) is called a groupoid with the object set \(X\) if all morphisms are isomorphisms.

For a finite small category \(M\) with the object set \(X\), as an analogous construction of groupoid algebras in [16, 2.5], an algebra \(k[M]\) over \(k\) is defined. This algebra is spanned by the morphisms of \(M\) over \(k\) as a vector space, and equipped with the product which is given by the composition if two morphisms are composable and by 0 otherwise. The identity element 1 in the algebra is given by 1 = \(\sum_{x \in X} 1_x\). The algebra \(k[M]\) becomes a weak bialgebra with the \(k\)-linear maps \(\Delta : k[M] \rightarrow k[M] \otimes k[M]\) and \(\varepsilon : k[M] \rightarrow k\) defined by \(\Delta(\alpha) = \alpha \otimes \alpha\) and \(\varepsilon(\alpha) = 1\) for all \(\alpha \in M\). If \(M\) is a groupoid, then the weak bialgebra \(k[M]\) is called a groupoid algebra. The groupoid algebra \(k[M]\) becomes a weak Hopf algebra with the above maps \(\Delta\) and \(\varepsilon\), and the \(k\)-linear map \(S : k[M] \rightarrow k[M]\) defined by \(S(\alpha) = \alpha^{-1}\) for all \(\alpha \in M\).

\textbf{Example 2.2} ([9, 1.2]). The following groupoid \(G\) over \(X = \{+, -\}\) is called the qubit groupoid:

\[
G = \{1_+, 1_-, + \xrightarrow{\alpha} -, - \xrightarrow{\alpha^{-1}} +\}.
\]

The corresponding quiver of the qubit groupoid is depicted as in the figure 1. The qubit groupoid algebra \(k[G]\) satisfies \(\varepsilon(1) = 2 \neq 1\), and hence it is not a Hopf algebra.

See [2] for more general construction and examples from quivers.

The target and source counital maps satisfy the following properties.
LEMMA 2.3. \( \varepsilon_1^2 = \varepsilon_1, \varepsilon_2^2 = \varepsilon_2, \varepsilon \circ \varepsilon_1 = \varepsilon = \varepsilon \circ \varepsilon_2. \)

(2) For all \( x, y \in H \)

(i) \( ((\text{id} \otimes \varepsilon_1) \circ \Delta)(x) = 1_1 x \otimes 1_2. \)

(ii) \( ((\varepsilon_2 \otimes \text{id}) \circ \Delta)(x) = 1_1 x \otimes 1_2. \)

(iii) \( \varepsilon_1(x) = x \iff \Delta(x) = 1_1 x \otimes 1_2. \)

(iv) \( \varepsilon_2(x) = x \iff \Delta(x) = 1_1 x \otimes 1_2. \)

(v) \( \varepsilon_1(x \varepsilon_2(y)) = \varepsilon_1(xy), \varepsilon_2(\varepsilon_1(x)) = \varepsilon_2(xy). \)

(vi) \( x = \varepsilon_1(x_{(1)}) x_{(2)} = x_{(1)} \varepsilon_2(x_{(2)}). \)

(vii) \( x \varepsilon_2(y) = \varepsilon_1(x_{(1)}) y_{(2)}, \varepsilon_2(x) y = y_{(1)} \varepsilon_1(x_{(2)}). \)

In particular, by (i) and (ii)

\[
\begin{align*}
1_1 (1_1 \otimes \varepsilon_1(1_2)) &= 1_1 \otimes 1_2 = \varepsilon_1(1_1) \otimes 1_2, \\
1_1 [1_1 \otimes 1_2 \otimes 1_2] &= 1_1 \otimes \varepsilon_1(1_2) \otimes 1_3, \\
1_1 (1_1 \otimes 1_3 \otimes 1_2) &= 1_1 \otimes \varepsilon_2(1_2) \otimes 1_3,
\end{align*}
\]

where \( \Delta(1) = 1_1 \otimes 1_2 \) and \( \Delta(1) = 1_1 \otimes 1_2. \)

See [5, Section 2] and [22, Section 4] for proofs of the above lemma.

Let \( H \) be a weak bialgebra over \( k \), and set \( H_t := \varepsilon_t(H), H_s := \varepsilon_s(H). \) Then an element of \( H_t \) and an element of \( H_s \) are commutative, and \( H_t \) and \( H_s \) are a left coideal subalgebra and a right coideal subalgebra of \( H \), respectively. The subalgebras \( H_t, H_s \) are called the target and source subalgebras of \( H \), respectively. By (2.3) we have

\[
\Delta(1) \in H_s \otimes H_t.
\]

From Lemma 2.3(iii), (iv) together with (2.4) and the fact that an element of \( H_t \) and an element of \( H_s \) are commutative, we see that for all \( x \in H, \; z \in H_t \) and \( y \in H_s \)

\[
\begin{align*}
\Delta(xz) &= x_{(1)} z \otimes x_{(2)}, \\
\Delta(zx) &= z x_{(1)} \otimes x_{(2)}, \\
\Delta(xy) &= x_{(1)} \otimes x_{(2)} y, \\
\Delta(yx) &= x_{(1)} \otimes y x_{(2)}.
\end{align*}
\]

In particular

\[
\begin{align*}
xz &= \varepsilon(x_{(1)}) z_{(2)}, \\
zx &= \varepsilon(z x_{(1)}) x_{(2)}, \\
xy &= x_{(1)} \varepsilon(x_{(2)}), \\
yx &= x_{(1)} \varepsilon(y x_{(2)}).
\end{align*}
\]

For a weak bialgebra \( H \), one can also consider two \( k \)-linear maps \( \varepsilon'_t, \varepsilon'_s : H \to H \) defined by

\[
\begin{align*}
\varepsilon'_t(x) &= \varepsilon(x_{(1)}) 1_{(2)}, \\
\varepsilon'_s(x) &= 1_{(1)} \varepsilon(1_{(2)} x).
\end{align*}
\]

The target and the source subalgebras of the three weak bialgebras \( H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon), H^{cop} = (H, \mu, \eta, \Delta^{cop}, \varepsilon), H^{copop} = (H, \mu^{op}, \eta, \Delta^{cop}, \varepsilon) \) are given

\[
\begin{align*}
\Delta(1) &= 1_1 \otimes 1_2 \quad \text{ and } \quad \Delta(1) = 1_1 \otimes 1_2.
\end{align*}
\]
by \((H^{\text{cop}})_t = H_t, (H^{\text{cop}})_s = H_s, (H^{\text{cop}})^t = H_s, (H^{\text{cop}})^s = H_t, (H^{\text{cop}}^{\text{cop}})_t = H_s, (H^{\text{cop}}^{\text{cop}})^s = H_t\). The target and the source counital maps of them are given by \((\varepsilon_{H^{\text{op}}})_t = \varepsilon'_t, (\varepsilon_{H^{\text{op}}})_s = \varepsilon'_s, (\varepsilon_{H^{\text{op}}})^t = \varepsilon'_t, (\varepsilon_{H^{\text{op}}})^s = \varepsilon'_s, (\varepsilon_{H^{\text{op}}^{\text{cop}}})_t = \varepsilon_s, (\varepsilon_{H^{\text{op}}^{\text{cop}}})_s = \varepsilon_t\). If \(S\) is the antipode of \(H\), then it is also of \(H^{\text{op}}^{\text{cop}}\).

The monoidal structure of the module category \(H \mathcal{M}\) for a weak bialgebra \(H\) is given as follows. Let \(V, W\) be two left \(H\)-modules, and define an \(H\)-action on the tensor product \(V \otimes W\) by

\[
(2.7) \quad x \cdot (v \otimes w) := (x(1) \cdot v) \otimes (x(2) \cdot w)
\]

for all \(v \in V, w \in W, x \in H\). By (WH1) we see that \((xy) \cdot \xi = x \cdot (y \cdot \xi)\) for all \(x, y \in H\) and \(\xi \in V \otimes W\). However, it is not certain whether \(1 \cdot \xi = \xi\). So, we consider the subspace

\[
(2.8) \quad V \otimes W := \Delta(1) \cdot (V \otimes W).
\]

This subspace becomes a left \(H\)-module with (2.7). For two left \(H\)-module maps \(f : V_1 \otimes W_1, g : V_2 \rightarrow W_2\), the tensor product \(f \otimes g\) satisfies \((f \otimes g)(V_1 \otimes V_2) \subseteq W_1 \otimes W_2\), and the restriction \(f \otimes g := f \otimes g|_{V_1 \otimes V_2}\) is a left \(H\)-module map. In this manner the monoidal structure of \(H \mathcal{M}\) is defined. The unit object in the monoidal category is given by the target subalgebra \(H_t\). Similarly, the monoidal structure of the comodule category \(\mathcal{M}^H\) is similarly defined. For more detail on the monoidal structures of \(H \mathcal{M}\) and \(\mathcal{M}^H\) see [5] and [18].

3. The direct sum construction of weak bialgebras and Hopf algebras

For two bialgebras \(A\) and \(B\), the direct sum \(A \oplus B\) becomes an algebra and a coalgebra. However it is not necessary to be a bialgebra. Whereas in weak case it is true as described in the following lemma.

**Lemma 3.1 (direct sum construction).** Let \(A = (A, \Delta_A, \varepsilon_A)\) and \(B = (B, \Delta_B, \varepsilon_B)\) be two weak bialgebras over \(k\), and set \(H = A \oplus B\) as a vector space. Then \(H\) has an weak bialgebra structure such that its algebra structure is given by the direct product and its coalgebra structure is given by the direct sum. The target and source counital maps \(\varepsilon_t\) and \(\varepsilon_s\) are given by the formulas: For \(x = a + b \in H\), where \(a \in A\) and \(b \in B\),

\[
\varepsilon_t(x) = (\varepsilon_A)_t(a) + (\varepsilon_B)_t(b),
\]

\[
\varepsilon_s(x) = (\varepsilon_A)_s(a) + (\varepsilon_B)_s(b),
\]

where \((\varepsilon_A)_t, (\varepsilon_A)_s\) are the target and source counital maps of \(A\), and \((\varepsilon_B)_t, (\varepsilon_B)_s\) are that of \(B\). The target and source subalgebras of \(H\) are \(H_t = A_t \oplus B_t\) and \(H_s = A_s \oplus B_s\), respectively.

If \(A, B\) are weak Hopf algebras with antipodes \(S_A, S_B\), then so is \(H\) with \(S\) defined by \(S(a + b) = S_A(a) + S_B(b)\) for all \(a \in A\) and \(b \in B\).

**Proof.** The claim follows from the simple equation \(ab = ba = 0\) for all \(a \in A\) and \(b \in B\). For example, it can be verified that the condition (WH1) is satisfied as follows. Let \(x, x'\) be elements in \(H\), and write as \(x = a + b, x' = a' + b'\) for some \(a, a' \in A, b, b' \in B\). Then

\[
\Delta(xx') = \Delta(aa' + bb') = \Delta_A(aa') + \Delta_B(bb') = \Delta_A(a)\Delta_A(a') + \Delta_B(b)\Delta_B(b')
\]
\[ \Delta_A(a) \Delta_B(b) = \Delta(x)\Delta(x'). \]

It can be also verified on the other conditions in a similar way. □

Chebel and Makhlouf [7] give a method to construct a weak bialgebra as an analogy of the method introduced by Kaplansky [10, Theorem 14]. Their construction can be viewed as a special version of a direct sum construction given in Lemma 3.1.

**Example 3.2 (Chebel and Makhlouf [7, Theorems 2.10 & 2.12])**. Let \( A = (A, \Delta_A, \varepsilon_A) \) be a bialgebra with identity element \( e \) over \( k \). Let \( 1 \) be a letter which is not contained in \( A \), and consider the vector space \( H := A \oplus k1 \). The product on \( A \) can be extended to the product on \( H \) as follows: For all \( a \in A \)
\[ 1 \cdot a = a = a \cdot 1, \quad 1 \cdot 1 = 1. \]

Moreover, we define \( k \)-linear maps \( \Delta : H \rightarrow H \otimes H, \varepsilon : H \rightarrow k \) by
\[ \Delta(a) = \Delta_A(a), \quad \varepsilon(a) = \varepsilon_A(a), \]
\[ \Delta(1) = (1 - e) \otimes (1 - e) + e \otimes e, \quad \varepsilon(1) = 2, \]
for all \( a \in A \). Then \( H \) is a weak bialgebra. If \( A \) is a Hopf algebra with antipode \( S_A \), then \( H \) is a weak Hopf algebra equipped with the antipode \( S : H \rightarrow H \) defined by
\[ S(a) = S_A(a), \quad S(1) = 1 \]
for all \( a \in A \). After [7] we call \( H \) the weak bialgebra obtained by the Kaplansky type construction from \( A \).

Applying the above construction to the Taft algebras we have a more concrete example.

**Example 3.3 (Taft’s weak bialgebra [7, Example 2.14])**. Let \( n \geq 2 \) be an integer, and \( k \) be a field which contains an \( n \)th root of unity \( \lambda \). Denoted by \( H_{n^2}(\lambda) \) is the \( n^2 \)-dimensional Taft algebra, that is
\[ H_{n^2}(\lambda) = \langle g, x \mid g^n = e, \ x^n = 0, \ xg = \lambda gx \rangle, \]
where \( e \) is the identity element. Then one can obtain a weak Hopf algebra \( H'_{n^2}(\lambda) \) with dimension \( n^2 + 1 \) by applying the Kaplansky type construction to \( H_{n^2}(\lambda) \).

A weak bialgebra \( H \) is said to be **indecomposable** if there are no weak bialgebras \( A, B \) such that \( H \cong A \oplus B \) as weak bialgebras. Similarly indecomposability can be defined for a weak Hopf algebra. Any finite-dimensional weak bialgebra can be decomposed into a direct sum of indecomposable ones up to isomorphism. The precise statement is as follows.

**Theorem 3.4.** Let \( H \) be a finite-dimensional weak bialgebra over \( k \).

1. There are finitely many indecomposable weak bialgebras \( H_i \) \((i = 1, \ldots, k)\) such that \( H = H_1 \oplus \cdots \oplus H_k \).
2. Assume that \( H_1 \oplus \cdots \oplus H_k = H = H'_1 \oplus \cdots \oplus H'_k \) for some indecomposable weak bialgebras \( H_1, \ldots, H_k \) and \( H'_1, \ldots, H'_k \). Then \( k = l \), and there is a permutation \( \sigma \) such that \( H'_i = H_{\sigma(i)} \) for all \( i = 1, \ldots, k \).
INDECOMPOSABILITY OF WEAK HOPF ALGEBRAS

PROOF. This theorem essentially follows from the existence and uniqueness for an algebra with respect to a direct sum decomposition of indecomposable algebras. (1) Let \( H = B_1 \oplus \cdots \oplus B_s \) be a decomposition into indecomposable algebras. We set \( \mathcal{B} := \{ B_1, \ldots, B_s \} \), and \( \langle S \rangle := \bigoplus_{B \in \mathcal{B}} B \) for \( S \subseteq \mathcal{B} \).

For each \( i \) let us consider the smallest subset \( S \subseteq \mathcal{B} \) satisfying \( \Delta(B_i) \subseteq \langle S \rangle \otimes \langle S \rangle \), and set \( S_i := S \cup \{ B_i \} \). Let \( \sim \) be the equivalence relation on \( \{1, \ldots, s\} \) generated by

\[ i \sim j \iff S_i \cap S_j \neq \emptyset. \]

If \( C_1, \ldots, C_m \) are all equivalence classes of \( \{1, \ldots, s\} \) with respect to \( \sim \), then \( \mathcal{B} \) is the disjoint union of

\[ T_1 := \bigcup_{i_1 \in C_1} S_{i_1}, \ldots, T_m := \bigcup_{i_m \in C_m} S_{i_m}. \]

Thus \( H = \langle T_1 \rangle \oplus \cdots \oplus \langle T_m \rangle \) as an algebra. Moreover one can easily check that \( \langle T_1 \rangle, \ldots, \langle T_m \rangle \) are weak bialgebras.

We show that \( \langle T_r \rangle \) is indecomposable as a weak bialgebra for all \( r \). Suppose that \( \langle T_r \rangle \) is not so. Then there are weak bialgebras \( H_1, H_2 \) such that \( \langle T_r \rangle = H_1 \oplus H_2 \). This decomposition can be viewed as an algebra. Applying the uniqueness of decomposition of \( \langle T_r \rangle \) into indecomposable algebras [23, Proposition 3.16] we see that \( H_1, H_2 \) are direct sums of several ones of \( B_1, \ldots, B_s \). Thus \( H_i \) can be expressed as \( H_i = \bigoplus_{B \in \mathcal{B}} B \) for some \( S_i \subseteq \mathcal{B} \). For any \( B_i \in S_1 \) we have \( \Delta(B_i) \subseteq \Delta(H_i) \subseteq H_1 \otimes H_1 \). Then \( S_1 \subseteq S_1 \cup \{ B_i \} = S_1 \) since \( S_1 \) is the union of \( \{ B_{i_1} \} \) and the smallest subset \( S \subseteq \mathcal{B} \) satisfying \( \Delta(B_{i_1}) \subseteq \langle S \rangle \otimes \langle S \rangle \). Similarly for any \( B_i \in S_2 \) we have \( S_{i_2} \subseteq S_2 \). If \( B_{i_1} \in S_1 \subseteq T_r \) and \( B_{i_2} \in S_2 \subseteq T_r \), then \( i_1, i_2 \in C_r \), and hence \( i_1 \sim i_2 \). This implies that \( S_{i_1} \cap S_{i_2} \neq \emptyset \), and so \( S_1 \cap S_2 \neq \emptyset \). This is a contradiction for the fact that \( H_1 \) and \( H_2 \) are in a direct sum in \( \langle T_r \rangle \). Thus \( \langle T_r \rangle \) is indecomposable as a weak bialgebra.

(2) Let \( H_i = B_1 \oplus \cdots \oplus B_s \) be a decomposition of \( H_1 \) into indecomposable algebras. By the uniqueness of decomposition into indecomposable algebras, for each \( B_i \) there is a unique \( j \in \{1, \ldots, l\} \) such that \( B_i \) is a direct summand of \( H_j \). Let us consider the case where \( s \geq 2 \), and suppose that \( B_1 \subseteq H_j^i \) and \( B_2 \subseteq H_j^i \). We show that \( i = j \).

For \( B_1, B_2 \) consider the subsets \( S_1, S_2 \) in \( \mathcal{B} = \{ B_1, \ldots, B_s \} \) defined as in the proof of Part (1). Define \( T_1, \ldots, T_m \) by introducing the equivalence relation \( \sim \) as in the proof of Part (1). Then \( H_1 = \langle T_1 \rangle \oplus \cdots \oplus \langle T_m \rangle \), and each \( \langle T_1 \rangle, \ldots, \langle T_m \rangle \) are weak bialgebras. By the indecomposability of \( H_1 \) as a weak bialgebra we have \( m = 1 \), that is, \( H_1 = \langle T_1 \rangle \). Thus \( B_1, B_2 \subseteq T_1 \), and \( S_1 \cap S_2 \neq \emptyset \).

Since \( B_1 \subseteq H_j^i \), we have \( \Delta(B_i) \subseteq H_j^i \otimes H_j^i \). This implies from the minimality of \( S_1 \) that \( S_1 \subseteq H_j^i \). Similarly, \( S_2 \subseteq H_j^i \). Thus for any \( B \in S_1 \cap S_2 \), we have \( B \subseteq S_1 \subseteq H_j^i \) and \( B \subseteq S_2 \subseteq H_j^i \). It follows that \( B \subseteq H_j^i \cap H_j^i \), and hence \( i = j \) since \( H_j^i \) and \( H_j^i \) form a direct sum in \( H \). Therefore, all \( B_i (i = 1, \ldots, s) \) are contained in the same \( H_j^i \). This shows that \( H_1 \subseteq H_j^i \). The same result holds for all \( H_1, \ldots, H_k \). Thus

\[ H_1 \subseteq H_1^i, \quad H_2 \subseteq H_2^i, \ldots, H_k \subseteq H_k^i \]

after suitable reordering of indices. In particular we have \( k \leq l \). Moreover, since \( H = H_1 \oplus \cdots \oplus H_k = H_1^i \oplus \cdots \oplus H_k^i \), it follows that \( k = l \).
Applying the same argument above, we see that each $H'_i$ ($i = 1, \ldots, k$) is contained in one of $H_1, \ldots, H_k$. This fact and (3.4) lead to the equation $H_i = H'_i$ ($i = 1, \ldots, k$). □

A finite-dimensional bialgebra is a typical example of indecomposable weak bialgebras.

Theorem 3.5. A finite-dimensional bialgebra is indecomposable as a weak bialgebra.

Proof. Suppose that a finite-dimensional bialgebra $H$ is decomposed as $H = A \oplus B$ for some weak bialgebras $A$ and $B$. By Lemma 3.1 then $H_i = A_i \oplus B_i$ as algebras, and hence $\dim H_i = \dim A_i + \dim B_i \geq 1 + 1 = 2$. This contradicts the fact $H_i = k1_H$. □

Example 3.6. Let $H = \mathbb{C}e_1 + \mathbb{C}e_2$ be a 2-dimensional weak bialgebra over the complex number field $\mathbb{C}$ whose product $\mu$, coproduct $\Delta$, counit $\varepsilon$ are given as follows:

$$
\mu(e_1, e_1) = e_1, \quad \mu(e_1, e_2) = \mu(e_2, e_1) = \mu(e_2, e_2) = e_2,
\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2, \quad \Delta(e_2) = e_2 \otimes e_2,
\varepsilon(e_1) = 2, \quad \varepsilon(e_2) = 1.
$$

Note that $e_1$ is the identity element. The weak bialgebra is the third one in the list in [7, Proposition 4.3] by Chebel and Makhlouf.

Consider the subspaces $A = \mathbb{C}(e_1 - e_2)$ and $B = \mathbb{C}e_2$. Then they are weak bialgebras isomorphic to the 1-dimensional Hopf algebra $\mathbb{C}$, and $H = A \oplus B$. Thus, the above weak bialgebra $H$ can be decomposed into the direct sum of two 1-dimensional Hopf algebras.

On the other hand, the second weak bialgebra $K = \mathbb{C}c_1 + \mathbb{C}c_2$ in the list in [7, Proposition 4.3] is isomorphic to the group algebra of $G = \mathbb{Z}/2\mathbb{Z}$. In fact, if we set $G = \langle g \rangle$, then an isomorphism $f : K \rightarrow \mathbb{C}[G]$ is given by $f(e_1) = 1, f(e_2) = (1 + g)/2$. In particular, $K$ is indecomposable.

Example 3.7. Chebel and Makhlouf [7] are determined the isomorphism classes of all weak bialgebras over the complex number field $\mathbb{C}$ of dimension \leq 3. According to their classification result, there are exactly 20 isomorphism classes in the 3-dimensional weak bialgebras. Let us denote the $n$th weak bialgebra by $(\#n)$ in the list [7, Proposition 4.5]. Then the 3-dimensional weak bialgebras except for $(\#8-10)$ are indecomposable as weak bialgebras, and $(\#8-10)$ can be decomposed into direct sums of indecomposable weak bialgebras as follows:

$$(\#8) = \mathbb{C} \oplus H_2, \quad (\#9) = \mathbb{C} \oplus \mathbb{C}[\mathbb{Z}/2\mathbb{Z}], \quad (\#10) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

Here, $H_2 = \mathbb{C}c_1 + \mathbb{C}c_2$ is the 2-dimensional weak bialgebra whose product, coproduct and counit are given by

$$
\mu(e_1, e_1) = e_1, \quad \mu(e_1, e_2) = \mu(e_2, e_1) = \mu(e_2, e_2) = e_2,
\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2,
\varepsilon(e_1) = \varepsilon(e_2) = 1.
$$

This weak bialgebra is the first one in the list in [7, Proposition 4.3].
which is spanned by the morphisms of the so-called interval category \(2\) also indecomposable since the former is isomorphic to the weak bialgebra \(\text{weak bialgebras since they are proper bialgebras. The weak bialgebras (#16) and (#11) also indecomposable since the former is isomorphic to the weak bialgebra which is spanned by the morphisms of the so-called interval category 2\).

(See Proposition 3.8 below), and the later is the dual of it.

\* The idempotents in \(H = (#8)\) are 0, \(e_1 - e_2\), \(e_2 - e_3\), \(e_3\), \(e_1 - e_3\), \(e_2\), \(e_1 - e_2 + e_3\), \(e_1\), and the identity element \(e_1\) is expressed as the sum of primitive idempotents \(e_1 - e_2\), \(e_2 - e_3\), \(e_3\). Thus as an algebra

\[
H = H(e_1 - e_2) \oplus H(e_2 - e_3) \oplus He_3 = C(e_1 - e_2) \oplus C(e_2 - e_3) \oplus Ce_3.
\]

Setting

\[
(3.5) \quad A_1 = C(e_1 - e_2), \quad A_2 = C(e_2 - e_3), \quad A_3 = Ce_3,
\]

we have a decomposition \(H = A_1 \oplus A_2 \oplus A_3\) into indecomposable algebras. Since

\[
\Delta(e_1 - e_2) = (e_1 - e_2) \otimes (e_1 - e_2),
\]

\[
\Delta(e_2 - e_3) = (e_2 - e_3) \otimes (e_2 - e_3) + (e_2 - e_3) \otimes e_3 + e_3 \otimes (e_2 - e_3),
\]

\[
\Delta(e_3) = e_3 \otimes e_3,
\]

it follows that \(A_1\), \(A_{23} := A_2 \oplus A_3\) are subcoalgebras of \(H\). Thus we have a decomposition \(H = A_1 \oplus A_{23}\) as a weak bialgebra. We see that the weak bialgebra \(A_{23}\) is isomorphic to the first weak bialgebra in the list in \([7, \text{Proposition 4.3}]\).

\* By a similar consideration above, it is shown that \(H = (#9)\) is decomposed as \(H = A_1 \oplus A_{23}\) as a weak bialgebra. In this case, since \(\Delta(e_2) = e_2 \otimes e_2\), \(\Delta(e_2 - e_3) = (e_2 - e_3) \otimes (e_2 - e_3) + e_3 \otimes e_3\), the weak bialgebra \(A_{23}\) is isomorphic to the second weak bialgebra in the list in \([7, \text{Proposition 4.3}]\). Thus, \(H \cong C \oplus C[Z/2Z]\) as weak bialgebras.

\* Let \(H = (#10)\). By a similar consideration above, we see that \(H\) is decomposed as \(H = A_1 \oplus A_2 \oplus A_3 \cong C \oplus C \oplus C\) as a weak bialgebra.

There are indecomposable weak bialgebras that are not proper bialgebras.

**Proposition 3.8.** Let \(M\) be a finite small category with the object set \(X\). Assume that the corresponding quiver \(Q_M\) is connected and has no multiple loops, namely, the identity morphism \(1_x\) is the unique morphism \(x \to x\) for all \(x \in X\).

Then the weak bialgebra \(H := k[M]\) is indecomposable as an algebra, and is also as a weak bialgebra.

**Proof.** Let \(e\) be an element in \(H\), and write as in the form

\[
e = \sum_{x \in X} a_x x + \sum_{\alpha \in M - X} a_\alpha \alpha
\]

for some \(a_x, a_\alpha \in k\). Since \(Q_M\) has no multiple loops, if \(xe = ex\) for all \(x \in X\), then \(a_\alpha = 0\) for all \(\alpha \in M - X\). Thus, if \(e\) is in the center \(Z(H)\) of \(H\), then it can be expressed as \(e = \sum_{x \in X} a_x x\). Let \(\alpha \in M - X\). Then \(\alpha e = a_{t(\alpha)} \alpha\), \(\alpha e = a_{s(\alpha)} \alpha\), and hence \(a_{t(\alpha)} = a_{s(\alpha)}\). Since \(Q_M\) is connected, for all \(x, y \in X\) there is a finite sequence \(x = x_0, x_1, \ldots, x_n = y\) such that \(x_{i-1}\) and \(x_i\) are related some morphism \(\alpha_i : x_{i-1} \to x_i\) in \(M\) for all \(i = 1, \ldots, n\). It follows that \(a_x\) does not depend on \(x\). Thus \(e \in Z(H)\) can be expressed as \(e = a \sum_{x \in X} x = a1\) for some \(a \in k\). So, if
\( e \in Z(H) \) is an idempotent, then there are only possibilities \( a = 0, 1 \). This shows that the algebra \( H \) is indecomposable. \( \square \)

**Example 3.9.** The qubit groupoid algebra is indecomposable as a weak bialgebra by Proposition 3.8.

We describe some properties of preserving under the direct sum construction.

Let \( H \) be a weak bialgebra over \( k \). An element \( \Lambda \in H \) is called a left integral of \( H \) if \( x\Lambda = \varepsilon_s(x)\Lambda \) for all \( x \in H \). Similarly, an element \( \Lambda \in H \) is called a right integral of \( H \) if \( \Lambda x = \Lambda \varepsilon_r(x) \) for all \( x \in H \). We denote the set of left integrals and right integrals by \( \mathcal{I}_L(H) \) and \( \mathcal{I}_R(H) \), respectively. An invertible element of \( g \in H \) is called group-like if \( \Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)g \otimes g \) is satisfied. We denote the set of group-like elements of \( H \) by \( G(H) \). The set \( G(H) \) becomes a group with respect to the product of \( H \) ([5, Definition 4.11], [15, Definition 4.1]). We note that if \( H \) has an antipode, then for any element \( g \in H \) satisfying \( \Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)g \otimes g \) the invertibility of \( g \) is equivalent to \( \varepsilon_s(g) = \varepsilon_r(g) = 1 \). The following is an easy consequence from definition and construction.

**Proposition 3.10.** Let \( A, B \) be two finite-dimensional weak bialgebras over \( k \), and consider the weak bialgebra \( H = A \oplus B \) obtained by the direct sum construction. Then

1. \( H \) is (co)semisimple if and only if both \( A, B \) are so.
2. \( \mathcal{I}_L(H) = \{ \Lambda_A + \Lambda_B \mid \Lambda_A \in \mathcal{I}_L(A), \Lambda_B \in \mathcal{I}_L(B) \} \), and the same equation holds for the set of right integrals.
3. \( G(H) \cong G(A) \times G(B) \) as groups.

4. The direct sum construction of weak Hopf algebras and quantum doubles

As in the Hopf algebra case [1], for any weak bialgebra \( H \), which is not necessary to be finite-dimensional, one can define the dual weak Hopf algebra as follows.

Let \( H = (H, \mu, \eta, \Delta, \varepsilon) \) be a weak bialgebra over \( k \), and set

\[ H^* := \{ p \in H^* \mid \dim(k[H]p) < \infty \}, \]

where \( H^* \) denotes the dual vector space of \( H \), and

\[ k[H] = \left\{ \sum_{x \in H} c_xx \mid c_x \in k, \ c_x = 0 \text{ except for finitely many } x \in H \right\}, \]

and \((\sum_{x \in H} c_xx)p\) is an element in \( H^* \) defined by

\[ \left( \left( \sum_{x \in H} c_xx \right)p \right)(h) = \sum_{x \in H} c_xx p(hx) \]

for all \( h \in H \). Then \( H^* \) becomes a weak bialgebra with the following product, identity element \( 1_{H^*} \), coproduct \( \Delta_{H^*} \), counit \( \varepsilon_{H^*} \): For all \( x, y \in H \) and \( p, q \in H^* \)

- \((pq)(x) = p(x_{(1)})q(x_{(2)})\),
- \(1_{H^*} = \varepsilon \) (\( = \) the counit of \( H \))
- \( p(1)(x)p(2)(y) = p(xy) \), where \( \Delta_{H^*}(p) = p(1) \otimes p(2) \),
- \( \varepsilon_{H^*}(p) = p(1) \).
The above weak bialgebra $H^\circ$ is called the dual weak bialgebra of $H$. If $H$ has an antipode $S$, then $H^\circ$ is also a weak Hopf algebra with the following $S_{H^\circ}$ as an antipode: For all $x \in H$ and $p \in H^\circ$

\begin{itemize}
  \item $(S_{H^\circ}(p), x) = (p, S(x))$,
\end{itemize}
where $(\ , \ )$ denotes the inner product defined by duality between $H$ and $H^*$. This weak Hopf algebra $H^\circ$ is called the dual weak Hopf algebra of $H$.

Let $H_1$, $H_2$ be two weak bialgebras over $k$. A $k$-linear map $f : H_1 \rightarrow H_2$ is called a weak bialgebra map if it is an algebra map and a coalgebra map. A bijective weak bialgebra map is called an isomorphism of weak bialgebras. A weak bialgebra map between weak Hopf algebras is called a weak Hopf algebra map. This definition is based on the fact that any weak bialgebra map $f : H_1 \rightarrow H_2$ between weak Hopf algebras $H_1$, $H_2$ is not only compatible with the target counital maps and the source counital maps of $H_1$, $H_2$, but also the antipodes of $H_1$, $H_2$. A bijective weak Hopf algebra map is called an isomorphism of weak Hopf algebras.

If a weak bialgebra $H$ is of finite dimension, then $H^* = H^\circ$. We regard as $H^* \otimes H^* = (H \otimes H)^*$ via the standard isomorphism $H^* \otimes H^* \rightarrow (H \otimes H)^*$ in linear algebra theory, and $\iota : H \rightarrow H^{**} = (H^*)^*$ be the canonical $k$-linear isomorphism defined by

\[(\iota(x))(p) = p(x)\]
for all $x \in H$, $p \in H^*$. Then, $\iota$ is an isomorphism of weak bialgebras, and if $H$ is a weak Hopf algebra, then $\iota$ is an isomorphism of weak Hopf algebras.

**Theorem 4.1.** Let $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ be a weak Hopf algebra over $k$. Then the antipode of the dual weak Hopf algebra $H^\circ$ is an anti-algebra map and an anti-coalgebra map.

**Proof.** The proof of what the antipode $S_{H^\circ}$ is an anti-algebra map and $\varepsilon \circ S = \varepsilon$ is the same in the proof of [5, Theorem 2.10]. We show that the equation $S_{H^\circ}(p_{(2)}) \otimes S_{H^\circ}(p_{(1)}) = \Delta_{H^\circ}(S_{H^\circ}(p))$ holds for all $p \in H^\circ$. For this it is enough to show that the images of the both sides under the injection $\pi : H^\circ \otimes H^\circ \rightarrow (H \otimes H)^*$, $\pi(p \otimes q) : x \otimes y \mapsto p(x)q(y)$ for all $p, q \in H^*$ and $x, y \in H$ coincide. It is verified as follows: For all $x, y \in H$ and $p \in H^\circ$

\[
\langle \pi(S_{H^\circ}(p_{(2)}) \otimes S_{H^\circ}(p_{(1)})), x \otimes y \rangle = \langle S_{H^\circ}(p_{(2)}), x \rangle \langle S_{H^\circ}(p_{(1)}), y \rangle \\
= \langle p_{(2)}, S(x) \rangle \langle p_{(1)}, S(y) \rangle \\
= \langle \pi(p_{(1)} \otimes p_{(2)}), S(y) \otimes S(x) \rangle \\
= \langle \pi(\Delta_{H^\circ}(p)), S(y) \otimes S(x) \rangle \\
= \langle p, S(y)S(x) \rangle \\
= \langle p, S(xy) \rangle \\
= \langle \pi(\Delta_{H^\circ}(S_{H^\circ}(p))), x \otimes y \rangle.
\]

\[\square\]

**Corollary 4.2.** Let $H$ be a finite-dimensional weak Hopf algebra over $k$. Then the antipode of $H$ is an anti-algebra map and an anti-coalgebra map.

**Proof.** Since $H$ is finite-dimensional, $H$ can be regarded as $H = H^{**}$. Then the statement follows from Theorem 4.1. \[\square\]
Theorem 4.3 (Böhm, Nill, Szlachányi [5, Theorem 2.10]). The antipode of a finite-dimensional weak Hopf algebra is bijective.

The concepts of quasitriangular weak Hopf algebras and weak ribbon algebras were introduced by Nikshych, Turaev and Vainerman [17]. A detailed study of quantum doubles of weak Hopf algebras is done in Chapter 6 in [17].

Let $H$ be a weak Hopf algebra over $k$, and $R \in \Delta^\text{cop}(1)(H \otimes H)\Delta(1)$. The pair $(H, R)$ and $R$ are called a quasitriangular weak Hopf algebra and a universal $R$-matrix of $H$, respectively, if the following four conditions are satisfied:

\begin{enumerate}[(Q1)]  
  \item There is an element $\overline{R} \in \Delta(1)(H \otimes H)\Delta^\text{cop}(1)$ such that $R\overline{R} = \Delta^\text{cop}(1)$ and $\overline{R}R = \Delta(1)$. 
  \item $\Delta^\text{cop}(x)R = R\Delta(x)$ for all $x \in H$. 
  \item $(\text{id} \otimes \Delta)(R) = R_{13}R_{12}$. 
  \item $(\Delta \otimes \text{id})(R) = R_{13}R_{23}$.
\end{enumerate}

Here, if we write as in the form $R = R^{(1)} \otimes R^{(2)}$, then $R_{13} = R^{(1)} \otimes 1 \otimes R^{(2)}$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$. We note that the element $\overline{R}$ in (Q1) is unique, and it satisfies $\Delta(x)\overline{R} = \overline{R}\Delta^\text{cop}(x)$ for all $x \in H$.

Proposition 4.4. Let $A, B$ be two weak Hopf algebras over $k$ of finite dimension, and $H = A \oplus B$ be the weak Hopf algebra obtained by the direct sum construction given in Lemma 3.1. If $R_A$ and $R_B$ are universal $R$-matrices of $A$ and $B$, respectively, then $R := R_A + R_B$ is a universal $R$-matrix of $H$. Conversely, any universal $R$-matrix of $H$ is given by the form $R_A + R_B$.

Proof. Since $ab = 0, ba = 0$ in $H$ for all $a \in A$ and $b \in B$,

\begin{equation}
\Delta^\text{cop}(1)(H \otimes H)\Delta(1) = \Delta^\text{cop}(1)(1_A)(A \otimes A)\Delta_A(1_A) + \Delta^\text{cop}(1)(1_B)(B \otimes B)\Delta_B(1_B).
\end{equation}

It follows that $R = R_A + R_B \in \Delta^\text{cop}(1)(H \otimes H)\Delta(1)$. It can be easily shown that $R$ satisfies the conditions (Q1), . . . , (Q4). Conversely, let $R$ be a universal $R$-matrix of $H$. By (4.2) $R$ can be written as in the form

$R = R_A + R_B \quad (R_A \in \Delta^\text{cop}(1)(A \otimes A)\Delta_A(1_A), \; R_B \in \Delta^\text{cop}(1)(B \otimes B)\Delta_B(1_B))$.

Similarly, $\overline{R} \in \Delta(1)(H \otimes H)\Delta^\text{cop}(1)$ in the condition (Q1) can be written as in the form

$\overline{R} = \overline{R}_A + \overline{R}_B \quad (\overline{R}_A \in \Delta_A(1_A)(A \otimes A)\Delta^\text{cop}_A(1_A), \; \overline{R}_B \in \Delta_B(1_B)(B \otimes B)\Delta^\text{cop}_B(1_B))$.

By using equations $ab = 0, ba = 0$ for all $a \in A$ and $b \in B$, we have $R_A\overline{R}_A = \Delta^\text{cop}_A(1_A)$, $R_B\overline{R}_B = \Delta^\text{cop}_B(1_B)$, $\overline{R}_AR_A = \Delta_A(1_A)$, $\overline{R}_BR_B = \Delta_B(1_B)$. Thus, $R_A$ and $R_B$ satisfy the condition (Q1). It can be easily shown that $R_A$ and $R_B$ satisfy the other three conditions. \hfill \Box

Example 4.5. Let $m, n \geq 2$, and suppose that $k$ contains an $m$th root of unity $\omega$ and an $n$th root of unity $\lambda$. Let us consider the Taft algebras $H_{m^2}(\omega)$ and $H_{n^2}(\lambda)$. Then, any universal $R$-matrix of the weak Hopf algebra $H := H_{m^2}(\omega) \oplus H_{n^2}(\lambda)$ is given by

$R_{H_{m^2}(\omega)} + R_{H_{n^2}(\lambda)}$,

where $R_{H_{m^2}(\omega)}$ and $R_{H_{n^2}(\lambda)}$ are universal $R$-matrices of $H_{m^2}(\omega)$ and $H_{n^2}(\lambda)$, respectively. In particular, if $m = n = 2$, then $\omega = \lambda = -1$, and $H_G(1)$ is Sweedler’s
4-dimensional Hopf algebra. By Radford [19] it is shown that if the characteristic
of \( k \) is not 2, then the universal \( R \)-matrices of \( H_4(-1) \) are given by
\[
R_\alpha = \frac{1}{2}(e \otimes e + g \otimes e + e \otimes g - g \otimes g) + \frac{\alpha}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x)
\]
for all \( \alpha \in k \). Thus, the universal \( R \)-matrices of the direct sum
\[
H = \left\langle e_1, g, x, e_2, h, y \right| \begin{array}{l}
g^2 = e_1, \ h^2 = e_2, \ x^2 = y^2 = 0, \\
xg = -gx, \ yg = -hy, \ e_1 + e_2 = 1 \\
ae_1 = e_1a = a, \ be_2 = e_2b = b \\
ab = ba = 0 \ (a \in \{e_1, g, x\}, \ b \in \{e_2, h, y\}) \end{array} \right\rangle
\]
of Sweedler’s 4-dimensional Hopf algebras are given by
\[
R_\alpha + R_\beta = \frac{1}{2}(e_1 \otimes e_1 + g \otimes e_1 + e_1 \otimes g - g \otimes g) + \frac{\alpha}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x) + \frac{1}{2}(e_2 \otimes e_2 + h \otimes e_2 + e_2 \otimes h - h \otimes h) + \frac{\beta}{2}(y \otimes y + y \otimes hy + hy \otimes hy - hy \otimes y)
\]
for all \( \alpha, \beta \in k \).

For two quasitriangular weak Hopf algebras \((A, R_A), (B, R_B)\), we set \((A, R_A) \oplus (B, R_B) := (A \oplus B, R_A + R_B)\), and call it the direct sum of \((A, R_A)\) and \((B, R_B)\).

Let \((H, R)\) be a weak quasitriangular Hopf algebra over \( k \). An element \( v \in H \)

is called a ribbon element of \((H, R)\) if the following three conditions are satisfied:

(RE1) \( v \) is invertible and central.
(RE2) \( \Delta(v) = R_{21}R(v \otimes v) \).
(RE3) \( S(v) = v \).

We denote by Rib\((H, R)\) the set of ribbon elements of \((H, R)\). A ribbon element

of \((H, R)\) can be characterized by some group-like element of \( H \) [17, Lemma A.1].

In straightforward computation we have:

**Proposition 4.6.** Let \( v_A, v_B \) be ribbon elements of two quasitriangular weak Hopf algebras \((A, R_A), (B, R_B)\), respectively. Then \( v = v_A + v_B \) is a ribbon element

of \((H, R) := (A \oplus B, R_A + R_B)\). Conversely, any ribbon element of \((H, R)\) is given

by the form \( v = v_A + v_B \) for some ribbon elements \( v_A, v_B \) of \((A, R_A), (B, R_B)\), respectively. That is, the map

\[
\text{Rib}(A, R_A) \times \text{Rib}(B, R_B) \longrightarrow \text{Rib}(H, R), \quad (v_A, v_B) \mapsto v_A + v_B
\]
is a bijection.

The direct sum construction is compatible with duals of weak bialgebras as follows.

**Lemma 4.7.** Let \( A, B \) be two weak bialgebras over \( k \) of finite dimension, and

\( H \) be the weak bialgebra obtained by the direct sum construction from \( A \) and \( B \). If

we regard as \( A^* \otimes B^* \subset H^* \) via the transposed maps \( ^t\pi_A, ^t\pi_B \) of canonical projections

\( \pi_A : H \longrightarrow A, \pi_B : H \longrightarrow B \) associated with the decomposition \( H = A \oplus B \), then

\( H^* = A^* \oplus B^* \) as weak bialgebras. More precisely, for all \( p, q \in H^* \)

- \( pq = ^t\pi_A(p|_A)^t\pi_A(q|_A) + ^t\pi_B(p|_B)^t\pi_B(q|_B) \),
\(1_{H^*} = \tau_{\pi_A}(\varepsilon_A) + \tau_{\pi_B}(\varepsilon_B),\)
\(
\Delta_{H^*}(p) = (\tau_{\pi_A} \otimes \tau_{\pi_A})(\Delta_{A^*}(p|_A)) + (\tau_{\pi_B} \otimes \tau_{\pi_B})(\Delta_{B^*}(p|_B)),
\)
\(
\varepsilon_{H^*}(p) = \varepsilon_{A^*}(p|_A) + \varepsilon_{B^*}(p|_B).
\)

If \(A\) and \(B\) are weak Hopf algebras, then for all \(p \in H^*\)
\(
S_{H^*}(p) = \tau_{\pi_A}(S_{A^*}(p|_A)) + \tau_{\pi_B}(S_{B^*}(p|_B)).
\)
Therefore, if \(H\) is indecomposable, then so is the dual weak bialgebra \(H^*\).

In the sequel, we examine a relationship between quantum doubles for weak Hopf algebras and the direct sum construction. We adopt here a version following Radford’s formulation of quantum doubles of Hopf algebras.

Let \(H\) be a weak bialgebra over \(k\). For elements \(x \in H\) and \(p \in H^*\), two actions \(x \leftarrow p \in H\) and \(p \rightarrow x \in H\) are defined by
\[
(4.3) \quad x \leftarrow p := \langle x(1), p \rangle x(2) = p(x(1))x(2),
\]
\[
(4.4) \quad p \rightarrow x := x(1)p \langle x(2) \rangle = x(1)p(x(2)).
\]
By dualizing these actions two actions \(p \leftarrow x \in H^*\) and \(x \rightarrow p \in H^*\) can be defined as
\[
(4.5) \quad (p \leftarrow x)(x') := p(xx'),
\]
\[
(4.6) \quad (x \rightarrow p)(x') := p(xx')
\]
for all \(x' \in H\).

The quantum double \(D(H)\) is defined by \(D(H) := H^* \otimes H / J\) as a vector space, where \(J = J_1 + J_2\), and
\[
(4.7) \quad J_1 = \text{Span}\{ p \otimes yx - p(\varepsilon \leftarrow y) \otimes x \mid y \in H_s, p \in H^*, x \in H \},
\]
\[
J_2 = \text{Span}\{ p \otimes zx - p(z \rightarrow \varepsilon) \otimes x \mid z \in H_t, p \in H^*, x \in H \}.
\]

The subspace \(J\) coincides with the kernel of the idempotent map
\[
(4.8) \quad F : H^* \otimes H \rightarrow H^* \otimes H, \quad F(p \otimes x) = (1_{(1)} \rightarrow p \leftarrow 1_{[1]}) \otimes 1_{(2)} \varepsilon_s(1_{[2]})x
\]
for all \(p \in H^*\) and \(x \in H\), and \(D(H)\) is isomorphic to the image of \(F\) as a vector space. This can be shown by using the fact that \(\varepsilon_s = 1_{(1)} \otimes \varepsilon_s(1_{(2)})\) is a Frobenius-separable idempotent of the source subalgebra \(H_s\) \([22, (4.13)]\). Since \(ab = 0 = ba\) for all \(a \in A\) and \(b \in B\), it follows that \(F(\pi_A(A^*) \otimes B) = F(\pi_B(B^*) \otimes A) = 0\). This implies that
\[
(4.9) \quad \langle \pi_A(p|_A) \otimes b \rangle = \langle \pi_B(p|_B) \otimes a \rangle = 0
\]
for all \(p \in H^*\), and therefore any element in \(D(H)\) can be expressed as
\[
(4.10) \quad \langle \pi_A(p|_A) \otimes a \rangle + \langle \pi_B(q|_B) \otimes b \rangle
\]
for some \(p, q \in H^*, a \in A, b \in B\).

Theorem 4.8. Let \(A, B\) be two weak Hopf algebras over \(k\) of finite dimension, and set \(H := A \oplus B\). Regard as \(D(A) \subset D(H)\) and \(D(B) \subset D(H)\) via the \(k\)-linear maps \(D(A) \rightarrow D(H)\) and \(D(B) \rightarrow D(H)\) induced from the injections \(t_{\pi_A} \otimes t_A : A^* \otimes A \rightarrow H^* \otimes H\) and \(t_{\pi_B} \otimes t_B : B^* \otimes B \rightarrow H^* \otimes H\), where \(\pi_A, \pi_B\) are projections, and \(t_A, t_B\) are inclusions. Then the quantum double \(D(H)\) is the direct sum of \(D(A)\) and \(D(B)\) as weak Hopf algebras: \(D(H) = D(A) \oplus D(B)\).
PROOF. First of all, we show that \( t_A \otimes \iota_A : A^* \otimes A \rightarrow H^* \otimes H \) induces an injection \( D(A) \rightarrow D(H) \). Let \( J_A \) be the subspace of \( A^* \otimes A \) defined by the sum space of (4.7) from \( A \) instead of \( H \), and we similarly define the subspace \( J_B \) of \( B^* \otimes B \). Then, \( J = J_A + J_B + A^* \otimes B + B^* \otimes A \) under the identifications \( A^* = t_A(A^*) \subset H^* \) and \( B^* = t_B(B^*) \subset H^* \). Since \( (t_A \otimes \iota_A)(J_A) = J_A \subset J \), the map \( t_A \otimes \iota_A : A^* \otimes A \rightarrow H^* \otimes H \) induces a \( k \)-linear map \( D(A) \rightarrow D(H) \). We write the map by \( f \):

\[
f(\alpha \otimes a) = [t_A(\alpha) \otimes \iota_A(a)]
\]

for all \( \alpha \in A^*, \ a \in A \). This map is injective since \( J_A = J \cap (A^* \otimes A) \).

The product of the quantum double \( D(H) := H^* \otimes H / J \) is given by

\[
[p \otimes x][q \otimes y] = (q(1), S^{-1}(x(3))) \langle q(3), x(1) \rangle [pq(2) \otimes x(2)y]
\]

for all \( p, q \in H^* \) and \( x, y \in H \). If we express as \( x = a + b, \ y = a' + b' \) for some \( a, a' \in A, \ b, b' \in B \), then

\[
[p \otimes x][q \otimes y] = [t_A(p) \otimes a] [t_A(q) \otimes a'] + [t_B(p) \otimes b] [t_B(q) \otimes b'].
\]

By using (4.9) we see that the identity element in \( D(H) \) is given by

\[
[\varepsilon(1)] = [t_A(\varepsilon_A) + t_B(\varepsilon_B)] (1_A + 1_B) = [t_A(\varepsilon_A) \otimes 1_A] + [t_B(\varepsilon_B) \otimes 1_B].
\]

Therefore \( D(H) \) is the direct product of \( D(A) \) and \( D(B) \) as an algebra. Moreover one can see that

\[
\Delta_{D(H)}([p \otimes x]) = \Delta_{D(A)}([p \otimes a]) + \Delta_{D(B)}([p \otimes b]),
\]

\[
\varepsilon_{D(H)}([p \otimes x]) = \varepsilon_{D(A)}([p \otimes a]) + \varepsilon_{D(B)}([p \otimes b]),
\]

\[
S_{D(H)}([p \otimes x]) = S_{D(A)}([p \otimes a]) + S_{D(B)}([p \otimes b])
\]

for all \( x = a + b \in H, \ a \in A, \ b \in B \). This implies that \( D(H) = D(A) \oplus D(B) \). □

EXAMPLE 4.9. (1) Let \( H = \langle 9 \rangle \). Then

\[H^* \cong C^* \oplus (C[Z/2Z])^* \cong C \oplus C[Z/2Z] = H,\]

\[D(H) \cong D(C) \oplus D(C[Z/2Z]) = C \oplus C[Z/2Z] \times Z/2Z.\]

It follows that \( D(H) \) is a 5-dimensional Taft’s weak algebra.

(2) For \( H = \langle 10 \rangle \), both of \( H^* \) and \( D(H) \) are isomorphic to \( H \).

REMARK 4.10. Part (1) has been shown by Zhang, Zhao and Wang [27]. In their paper, the ribbon elements of \( (D(H), \mathcal{R}) \) are determined. This result can be confirmed by Proposition 4.6.

5. Categorical interpretation of the direct sum construction and its generalizations

A \( k \)-linear monoidal category \( \mathcal{C} \) is said to be indecomposable if there are no \( k \)-linear monoidal categories \( \mathcal{C}_1, \mathcal{C}_2 \) such that \( \mathcal{C} \cong \mathcal{C}_1 \times \mathcal{C}_2 \) as \( k \)-linear monoidal categories.

PROPOSITION 5.1. Let \( H \) be a weak bialgebra over \( k \). If \( H \) is decomposable, then

(1) the \( k \)-linear monoidal categories \( H \mathcal{M} \) and \( H \mathcal{M} \) are decomposable.
(2) the $k$-linear monoidal categories $M^H$ and $M^H$ are decomposable.

**Proof.** Suppose that $H$ is decomposed as $H = A \oplus B$ for some two weak bialgebras $A$ and $B$.

(1) Any left $H$-module $X$ is decomposed as $X = (1_A \cdot X) \oplus (1_B \cdot X)$. This decomposition gives rise to identical equivalences $H^M \simeq A^M \times B^M$ and $H^M \simeq A^M \times B^M$ as $k$-linear monoidal categories.

(2) Any right $H$-comodule $(X, \rho)$ is decomposed as $(X, \rho) = ((\varepsilon_A \circ \pi_A) \cdot X, (\text{id} \otimes \pi_A) \circ \rho |_{(\varepsilon_A \circ \pi_A) \cdot X}) \oplus ((\varepsilon_B \circ \pi_B) \cdot X, (\text{id} \otimes \pi_B) \circ \rho |_{(\varepsilon_B \circ \pi_B) \cdot X})$, where $\pi_A : H \rightarrow A$, $\pi_B : H \rightarrow B$ are the projections associated to the direct sum decomposition $H = A \oplus B$. This decomposition gives rise to identical equivalences $M^H \simeq M^A \times M^B$ and $M^H \simeq M^A \times M^B$ as $k$-linear monoidal categories. \Box

The converse of the above proposition is also true. Namely, we have:

**Theorem 5.2.** Let $H$ be a finite-dimensional weak bialgebra over $k$. Then $H$ is indecomposable as a weak bialgebra if and only if the module category $H^M$ is indecomposable as a $k$-linear monoidal category.

The theorem can be proved by using the Tannaka-Krein reconstruction theorem of a bialgebra established by Ulbrich [25] and Schauenburg [21, Theorem 5.4] and by using a reconstruction theorem of a weak bialgebra map, which is a generalization of the reconstruction theorem of a bialgebra map [13, Theorem 2.2], [20, Theorem 2.2]. See [26] for detail of the proof of Theorem 5.2.

Let us consider a generalization of direct sum construction for weak bialgebras. For this we use the concept of weak bicomodule algebras which is introduced by Bhöm [4, p.4689].

**Definition 5.3.** Let $H, K$ be two weak bialgebras over $k$, and $A$ be an algebra over $k$.

(1) Let $\lambda_A$ be a left $K$-coaction. The pair $(A, \lambda_A)$ is called a left weak $K$-comodule algebra if the following two conditions are satisfied for all $a, b \in A$:

\begin{align*}
\text{(LWCA1)} & \quad \lambda_A(ab) = \lambda_A(a)\lambda_A(b). \\
\text{(LWCA2)} & \quad (1_K \otimes a)\lambda_A(1_A) = (\varepsilon_s \otimes \text{id})(\lambda_A(a)).
\end{align*}

Under the condition (LWCA1) it is known that the following each condition is equivalent to (LWCA2).

\begin{align*}
\text{(i)} & \quad (\text{id} \otimes \lambda)(\lambda(1_A)) = 1_{(1)} \otimes (1_A)(1_{(2)} \otimes (1_A)(0)), \\
\text{(ii)} & \quad (\text{id} \otimes \lambda)(\lambda(1_A)) = 1_{(1)} \otimes 1_{(2)}(1_A)(-1) \otimes (1_A)(0), \\
\text{(iii)} & \quad \varepsilon_s(a_{(-1)}) \otimes a_{(0)} = (1_A)(-1) \otimes a(1_A)(0) \text{ for all } a \in A.
\end{align*}

Some more useful equivalent conditions for that $A$ is a weak $K$-comodule algebra can be found in [6, Proposition 4.10]. We remark that a left weak $K$-comodule algebra $(A, \lambda_A)$ is characterized as an algebra in the left comodule category $^H M$.

(2) Let $\rho_A$ be a right $H$-coaction. The pair $(A, \rho_A)$ is called a right weak $H$-comodule algebra if the following two conditions are satisfied for all $a, b \in A$:

\begin{align*}
\text{(RWCA1)} & \quad \rho_A(ab) = \rho_A(a)\rho_A(b). \\
\text{(RWCA2)} & \quad \rho_A(1_A)(a \otimes 1_H) = (\text{id} \otimes \varepsilon_t)(\rho_A(a)).
\end{align*}
(3) Let \((A, \lambda_A)\) and \((A, \rho_A)\) be a left weak \(K\)-comodule algebra and a right weak \(H\)-comodule algebra, respectively. The triple \((A, \lambda_A, \rho_A)\) is called a weak \((K, H)\)-comodule algebra if
\[
\begin{align*}
\lambda_A \circ \lambda_A & = \lambda_A, \\
\lambda_A \circ \rho_A & = \lambda_A, \\
(\lambda_A(1) \otimes 1)(1 \otimes \rho_A(1)) & = (1 \otimes \rho_A(1))(\lambda_A(1) \otimes 1) = (\lambda_A(1) \otimes \lambda_A(1)).
\end{align*}
\]

**Lemma 5.4.** Let \(H\) be a weak bialgebra over \(k\), and \((A, \lambda_A)\) be a left weak \(H\)-comodule algebra. Then

1. For a left \(H\)-module \(X\) and a left \(A\)-module \(M\), a left \(A\)-action on \(X \otimes M\) is defined by
   \[
a \cdot (x \otimes m) = a_{(-1)} \cdot x \otimes a_{(0)} \cdot m
   \]
   for all \(a \in A\), \(x \in X\), \(m \in M\). Then \(X \otimes M := \lambda_A(1_A) \cdot (X \otimes M)\) is a left \(A\)-module.
2. For a left \(H\)-module map \(f : X \rightarrow Y\) and a left \(A\)-module map \(\varphi : M \rightarrow N\), the \(k\)-linear map \(f \otimes \varphi\) induces a left \(A\)-module map \(f \otimes \varphi : X \otimes M \rightarrow Y \otimes N\).
3. The correspondences of (1) and (2) give rise to a \(k\)-linear covariant functor \(\otimes : A \otimes M \rightarrow A \otimes M\), and it is a left action of \(A\) on \(A \otimes M\) in the following sense: For \(X, Y \in A \otimes M\) and \(M \in A \otimes M\), there are natural left \(A\)-module isomorphisms
   \[
   X \otimes (Y \otimes M) \cong (X \otimes Y) \otimes M, \quad H_t \otimes M \cong M.
   \]
   where \(X \otimes Y\) stands for the tensor product of \(X\) and \(Y\) in \(A \otimes M\).
4. Let \(B\) be an algebra over \(k\). In (1) if \(M\) is an \((A, B)\)-module, then so is \(X \otimes M\), and in (2) if \(\varphi : M \rightarrow N\) is an \((A, B)\)-module map, then so is \(f \otimes \varphi\), too. Therefore, we have a covariant functor \(\otimes : A \otimes M \rightarrow A \otimes M\), for an \((A, B)\)-module \(M\) the natural isomorphisms \(X \otimes (Y \otimes M) \cong (X \otimes Y) \otimes M\), \(H_t \otimes M \cong M\) in (3) are \((A, B)\)-module isomorphisms.

**Proof.** Parts (1) and (2) are immediately verified.

(3) It is obvious that the usual \(k\)-linear isomorphism
\[
a_{X,Y,M} : (X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M), \quad (x \otimes y) \otimes m \mapsto x \otimes (y \otimes m)
\]
induces a \(k\)-linear isomorphism \((X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)\). By \((\Delta \otimes \text{id}_A) \circ \lambda_A = (\text{id} \otimes \lambda_A) \circ \lambda_A\) the isomorphism \((X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)\) is a left \(A\)-module map. We denote it by the same symbol \(a_{X,Y,M}\).

Consider the \(k\)-linear map \(l_M : H_t \otimes M \rightarrow M\) defined by
\[
l_M(z \otimes m) = \varepsilon(z)m
\]
for all \(z \in H_t\), \(m \in M\). We denote the restriction of \(l_M\) to \(H_t \otimes M\) by \(l_M\), again. Then the map \(l_M : H_t \otimes M \rightarrow M\) is a left \(A\)-module map, and it is natural on \(M\). Moreover, it is an isomorphism since the inverse of \(l_M\) is given by the map \(l_M^l : M \rightarrow H_t \otimes M\) defined by
\[
l_M^l(m) = (1_A)_{(-1)} \cdot 1_H \otimes (1_A)_{(0)} \cdot m = \varepsilon((1_A)_{(-1)}) \otimes (1_A)_{(0)} \cdot m
\]
for all \(m \in M\).
(4) Consider the case where $M$ is an $(A, B)$-module in (1), and define a right $B$-action on $X \otimes M$ by
\[(x \otimes m) \cdot b = x \otimes (m \cdot b)\]
for all $x \in X$, $m \in M$, $b \in B$. Equipped with this right action $X \otimes M$ is an $(A, B)$-module. It is easily verified that the rest conditions hold. \qed

The right comodule algebra version of the above lemma also holds. In this case, instead of $(A, \lambda_A)$ a right weak $H$-comodule algebra $(B, \rho_B)$ is used, and for a left $H$-module $X$ and a left $B$-module $M$, a right $B$-action on $M \otimes X$ is defined by
\[b \cdot (m \otimes x) = b_{(0)} \cdot m \otimes b_{(1)} \cdot x\]
for all $b \in B$, $x \in X$, $m \in M$. Then $M \otimes X := \rho_B(1_B) \cdot (M \otimes X)$ is a left $B$-module. As a similar manner in Lemma 5.4 a $k$-linear covariant functor $\otimes : B^M \times h^M \to B^M$ is obtained, and it is a right action of $h^M$ on $B^M$:
\[(M \otimes X) \otimes Y \cong M \otimes (X \otimes Y), \quad M \otimes H_t \cong M\]
for $X, Y \in h^M$ and $M \in B^M$.

Given two weak bialgebras $H, K$, a weak $(H, K)$-comodule algebra $A = (A, \lambda_A, \rho_A)$ and a weak $(K, H)$-comodule algebra $B = (B, \lambda_B, \rho_B)$, we will make $L = H \oplus A \oplus B \oplus K$ be a weak bialgebra. By using the equations (i), (ii) in Definition 5.3 (1) the following proposition, which gives a generalization of the direct sum construction, can be verified.

**Proposition 5.5.** Let $H, K$ be two weak bialgebras over $k$, and $A = (A, \lambda_A, \rho_A)$, $B = (B, \lambda_B, \rho_B)$ be a weak $(H, K)$-comodule algebra and $(K, H)$-comodule algebra, respectively. Let $\Delta_{A,B} : H \to A \otimes B$, $\Delta_{B,A} : K \to B \otimes A$ be $k$-linear maps preserving products. Set $L := H \oplus A \oplus B \oplus K$, and consider it as the algebra obtained by the direct product of $H, A, B, K$. Define $k$-linear maps $\Delta_L : L \to L \otimes L$ and $\varepsilon_L : L \to k$ by
\[
\begin{align*}
\Delta_L(x) &= \Delta_H(x) + \Delta_{A,B}(x), & \varepsilon_L(x) &= \varepsilon_H(x), \\
\Delta_L(a) &= \lambda_A(a) + \rho_A(a), & \varepsilon_L(a) &= 0, \\
\Delta_L(b) &= \lambda_B(b) + \rho_B(b), & \varepsilon_L(b) &= 0, \\
\Delta_L(y) &= \Delta_K(y) + \Delta_{B,A}(y), & \varepsilon_L(y) &= \varepsilon_K(y)
\end{align*}
\]
for all $x \in H$, $a \in A$, $b \in B$, $y \in K$. Then $(L, \Delta_L, \varepsilon_L)$ is a weak bialgebra if and only if the 16 conditions, which consist of the following 8 conditions (1), \ldots, (8) and the more 8 conditions obtained from (1), \ldots, (8) by interchanging $A, H$ with $B, K$, respectively, are satisfied.

1. $(\Delta_{A,B} \otimes \text{id}_H) \circ \Delta_H = (\text{id}_A \otimes \rho_B) \circ \Delta_{A,B}$.
2. $(\lambda_A \otimes \text{id}_B) \circ \Delta_{A,B} = (\text{id}_H \otimes \Delta_{A,B}) \circ \Delta_H$.
3. $(\rho_A \otimes \text{id}_B) \circ \Delta_{A,B} = (\text{id}_A \otimes \lambda_B) \circ \Delta_{A,B}$.
4. $(\Delta_{A,B}(1_H) \otimes 1_H)(1_A \otimes \rho_B(1_B)) = (1_A \otimes \rho_B(1_B))(\Delta_{A,B}(1_H) \otimes 1_H) = (\Delta_{A,B} \otimes \text{id}_H)(\Delta_H(1_H))$.
5. $(\lambda_A(1_A) \otimes 1_B)(1_H \otimes \Delta_{A,B}(1_H)) = (1_H \otimes \Delta_{A,B}(1_H))(\lambda_A(1_A) \otimes 1_B) = (\lambda_A \otimes \text{id}_B)(\Delta_{A,B}(1_H))$.
6. $(\rho_A(1_A) \otimes 1_B)(1_A \otimes \lambda_B(1_B)) = (1_A \otimes \lambda_B(1_B))(\rho_A(1_A) \otimes 1_B) = (\rho_A \otimes \text{id}_B)(\Delta_{A,B}(1_H))$.
7. $(\Delta_{A,B} \otimes \text{id}_A) \circ \lambda_A = (\text{id}_A \otimes \Delta_{B,A}) \circ \rho_A$. 

obtained from them by interchanging condition (WH2) is equivalent to the conditions (4), (5), (6), (8) and the conditions satisfied. Thanks to the conditions (i), (ii) in Definition 5.3(1), we see also that the condition \((\lambda_1, \lambda_2)\) becomes a weak Hopf algebra with the following antipode \(S_L\): For all \(x \in H\), \(a \in A\), \(b \in B\), \(y \in K\)

\[
S_L(x + a + b + y) = S_H(x) + S_K(y).
\]

**Proof.** The conditions (WH1) and (WH3) for \(L\) are satisfied by the assumption immediately. Coassociativity of \(\Delta_L\) is equivalent to the conditions (1), (2), (3), (7) and the conditions obtained from them by interchanging \(A\), \(H\) with \(B\), \(K\), respectively. The condition \((\varepsilon_L \otimes \text{id}) \circ \Delta_L = \text{id}_L = (\text{id}_L \otimes \varepsilon_L) \circ \Delta_L\) is always satisfied. Thanks to the conditions (i), (ii) in Definition 5.3(1), we see also that the condition (WH2) is equivalent to the conditions (4), (5), (6), (8) and the conditions obtained from them by interchanging \(A, H\) with \(B, K\), respectively.

**Remark 5.6.** The conditions (1), (2) are equivalent to what \(\Delta_{A, B}\) is an \((H, H)\)-comodule map.

Two maps \((\lambda_A)_t, (\rho_B)_s\) do not preserve products, but they satisfy the following formulas.

**Lemma 5.7.** Let \(H, K\) be two weak bialgebras over \(k\), and \(A = (A, \lambda_A, \rho_A), B = (B, \lambda_B, \rho_B)\) be a weak \((H, K)\)-comodule algebra and a weak \((K, H)\)-comodule algebra, respectively. Define \(k\)-linear maps \((\lambda_A)_t : H \rightarrow A, (\rho_A)_s : K \rightarrow A, (\lambda_B)_t : K \rightarrow B, (\rho_B)_s : H \rightarrow B\) as follows: For all \(x \in H\) and \(y \in K\)

\[\begin{align*}
(\lambda_A)_t(x) &= \varepsilon_H ((1_A)^H_{(0)} x^t(1_A)^H_{(1)}), \\
(\rho_A)_s(y) &= (1_A)^K_{(0)} \varepsilon_K (y(1_A)^K_{(1)}), \\
(\lambda_B)_t(y) &= \varepsilon_K ((1_B)^K_{(1)} y^t(1_B)^K_{(0)}), \\
(\rho_B)_s(z) &= (1_B)^H_{(0)} \varepsilon_H (x(1_B)^H_{(1)}).
\end{align*}\]

Then for all \(x, x', y \in K\)

\[
(\lambda_A)_t(x)(\lambda_A)_t(x') = (\lambda_A)_t(\left(\varepsilon_H(\rho_A)_s(x)(\rho_B)_s(x')\right)) = (\rho_B)_s(x(\varepsilon_H(\rho_A)_s(x'))),
\]

\[
(\lambda_B)_t(y)(\lambda_B)_t(y') = (\lambda_B)_t(\left((\varepsilon_K(\rho_A)_s(y)(\rho_B)_s(y')\right)) = (\rho_A)_s(y(\varepsilon_K(\rho_B)_s(y'))).
\]

**Proof.** The first equation is derived as follows.

\[
(\lambda_A)_t(x)(\lambda_A)_t(x') = \varepsilon_H((1_A)^H_{(0)} x^t)_t \varepsilon_H((1_A)^H_{(1)} x')^t_0 (1_A)^H_{(0)} (1_A)^H_{(1)}
\]

\[
= \varepsilon_H((1_A)^H_{(0)} x^t) \varepsilon_H((1_A)^H_{(1)} x') (1_A)^H_{(0)} (1_A)^H_{(1)}
\]

\[
= \varepsilon_H((1_A)^H_{(0)} x^t) \varepsilon_H((1_A)^H_{(1)} x')^t_0 (1_A)^H_{(0)} (1_A)^H_{(1)}
\]

\[
= \varepsilon_H((1_A)^H_{(1)} x^t) \varepsilon_H((1_A)^H_{(2)} x')^t_0 (1_A)^H_{(0)} (1_A)^H_{(1)}
\]

\[
= \varepsilon_H((1_A)^H_{(1)} x^t) (1_H)^H_{(2)} x' (1_A)^H_{(0)} (1_A)^H_{(1)}
\]

\[
= (\lambda_A)_t(\left((\varepsilon_H(\rho_A)_s(x)(\rho_B)_s(x')\right))
\]

Other equations are derived in a similar way.

\(\square\)
Lemma 5.8. Under the same situation in Proposition 5.5, the target and source counital maps \((\varepsilon_L)_t\) and \((\varepsilon_L)_s\) of the weak bialgebra \(L = H \oplus A \oplus B \oplus K\) are given by
\[
(\varepsilon_L)_t(x + a + b + y) = ((\varepsilon_H)_t + (\lambda_A)_t)(x) + ((\lambda_B)_t + (\varepsilon_K)_t)(y),
\]
\[
(\varepsilon_L)_s(x + a + b + y) = ((\varepsilon_H)_s + (\rho_B)_s)(x) + ((\rho_A)_s + (\varepsilon_K)_s)(y)
\]
for all \(x \in H, a \in A, b \in B, y \in K\), where \((\lambda_A)_t, (\rho_A)_s, (\lambda_B)_s, (\rho_B)_s\) are the same defined in Lemma 5.7.

Theorem 5.9. Let \(H, K\) be two weak bialgebras, and \(A = (A, \lambda_A, \rho_A)\) and \(B = (B, \lambda_B, \rho_B)\) be a weak \((H, K)\)-comodule algebra and a weak \((K, H)\)-comodule algebra, respectively. Let \(\Delta_{A,B} : H \rightarrow A \otimes B, \Delta_{B,A} : K \rightarrow B \otimes A\) be an \((H, H)\)-comodule map and \(A, B, K\)-comodule map, respectively. Consider the \(k\)-linear category \(\begin{pmatrix} H \mathcal{M} & A \mathcal{M} \\ B \mathcal{M} & K \mathcal{M} \end{pmatrix}\) of the Cartesian product of \(k\)-linear categories \(H \mathcal{M}, A \mathcal{M}, B \mathcal{M}, K \mathcal{M}\). Then this \(k\)-linear category becomes a \(k\)-linear monoidal category equipped with the tensor product \(\otimes\) defined as follows: For \(X_1, X_2 \in H \mathcal{M}, M_1, M_2 \in A \mathcal{M}, N_1, N_2 \in B \mathcal{M}, Y_1, Y_2 \in K \mathcal{M}\)
\[
\begin{pmatrix} X_1 & M_1 \\ N_1 & Y_1 \end{pmatrix} \otimes \begin{pmatrix} X_2 & M_2 \\ N_2 & Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \otimes X_2 + M_1 \otimes N_2 & X_1 \otimes M_2 + M_1 \otimes Y_2 \\ N_1 \otimes X_2 + Y_1 \otimes N_2 & N_1 \otimes M_2 + Y_1 \otimes Y_2 \end{pmatrix},
\]
where the symbol \(+\) in the right-hand side means the direct product, and
\[
M_1 \otimes N_2 := \Delta_{A,B}(1_H) \cdot (M_1 \otimes N_2),
\]
\[
N_1 \otimes M_2 := \Delta_{B,A}(1_K) \cdot (N_1 \otimes M_2).
\]
The tensor product of morphisms is defined in the same way, and \(I := \begin{pmatrix} H_t & O \\ O & K_t \end{pmatrix}\) is a unit object in the monoidal category.

Furthermore, for the weak bialgebra \(L := H \oplus A \oplus B \oplus K\) defined in Proposition 5.5, we have \(L \mathcal{M} \simeq \begin{pmatrix} H \mathcal{M} & A \mathcal{M} \\ B \mathcal{M} & K \mathcal{M} \end{pmatrix}\) as \(k\)-linear monoidal categories.

Proof. The monoidal category structure of \(\begin{pmatrix} H \mathcal{M} & A \mathcal{M} \\ B \mathcal{M} & K \mathcal{M} \end{pmatrix}\) is given as follows.

(i) For three objects \(V_1, V_2, V_3\), there is a natural isomorphism \(a_{V_1, V_2, V_3} : (V_1 \otimes V_2) \otimes V_3 \rightarrow V_1 \otimes (V_2 \otimes V_3)\) induced from shifting parentheses by component-wise since \(\otimes\) in \(\begin{pmatrix} H \mathcal{M} & A \mathcal{M} \\ B \mathcal{M} & K \mathcal{M} \end{pmatrix}\) is defined by the same rule with matrix product. The isomorphisms \(\{a_{V_1, V_2, V_3}\}\) obviously satisfy the pentagon identity.

(ii) For an object \(V := \begin{pmatrix} X & M \\ N & Y \end{pmatrix}\), we have
\[
I \otimes V = \begin{pmatrix} H_t \otimes X \\ K_t \otimes N \end{pmatrix} \otimes \begin{pmatrix} H_t \otimes M \\ K_t \otimes Y \end{pmatrix}, \quad V \otimes I = \begin{pmatrix} X \otimes H_t \\ N \otimes H_t \end{pmatrix} \otimes \begin{pmatrix} M \otimes K_t \\ Y \otimes K_t \end{pmatrix}.
\]
Let \(l_X : H_t \otimes X \rightarrow X, l_M : H_t \otimes M \rightarrow M, l_N : K_t \otimes N \rightarrow N, l_Y : K_t \otimes Y \rightarrow Y\) be the isomorphisms of left annihilation and \(r_X : X \otimes H_t \rightarrow X, r_M : M \otimes K_t \rightarrow M, r_N : N \otimes H_t \rightarrow N, r_Y : Y \otimes K_t \rightarrow Y\) be the isomorphisms of right annihilation. Then
\[
l_V := \begin{pmatrix} l_X \\ l_N \end{pmatrix} : I \otimes V \rightarrow V, \quad r_V := \begin{pmatrix} r_X & r_M \\ r_N & r_Y \end{pmatrix} : V \otimes I \rightarrow V.
\]
are natural isomorphisms. For two objects $V_1, V_2$, we have $(\text{id}_{V_1} \otimes I_{V_2}) \circ a_{V_1, I, V_2} = (r_{V_1} \otimes \text{id}_{V_2})$ by using the triangle identity in component-wise. Thus, $(H_{B, M}^M A_{M}^M)_{B, M}$ is a monoidal category.

We will show that this monoidal category is equivalent to $L \mathcal{M}$. Let $V$ be a left $L$-module. Since the identity element of $L$ is $1_L = 1_H + 1_A + 1_B + 1_K$, the left $L$-module $V$ is decomposed as $V = (1_H \cdot V) \oplus (1_A \cdot V) \oplus (1_B \cdot V) \oplus (1_K \cdot V)$. Define a covariant functor $F : L \mathcal{M} \to \left( \begin{array}{cc} H_{B, M}^M A_{M}^M \\ B_{M}^M K_{M}^M \end{array} \right)$ as follows:

- $F(V) = \left( \begin{array}{cc} 1_H \cdot V \\ 1_A \cdot V \\ 1_B \cdot V \\ 1_K \cdot V \end{array} \right)$ for an object $V$,
- $F(f) = \left( \begin{array}{cc} f_{1_H \cdot U} \\ f_{1_A \cdot U} \\ f_{1_B \cdot U} \\ f_{1_K \cdot U} \end{array} \right)$ for a morphism $f : U \to V$.

The functor $F$ is $k$-linear and gives an equivalence of $k$-linear categories. Moreover, $F$ preserves tensor product. In fact, let $U, V$ be two left $L$-modules, and consider the left $L$-module $U \otimes V$. Setting $X_1 := 1_H \cdot U$, $M_1 := 1_A \cdot U$, $N_1 := 1_B \cdot U$, $Y_1 := 1_K \cdot U$ and $X_2 := 1_H \cdot V$, $M_2 := 1_A \cdot V$, $N_2 := 1_B \cdot V$, $Y_2 := 1_K \cdot V$, we have

$$U \otimes V = X_1 \otimes X_2 + M_1 \otimes N_2 + X_1 \otimes M_2 + M_1 \otimes Y_2 + N_1 \otimes X_2 + Y_1 \otimes N_2 + N_1 \otimes M_2 + Y_1 \otimes Y_2.$$ This implies that $F(U \otimes V) = F(U) \otimes F(V)$.

Next, we compute the image of the unit object $L_I$ under $F$. Set

$$(H \oplus A)_I := \text{Im}( (\varepsilon_H)_I + (\lambda_A)_I), \quad (K \oplus B)_I := \text{Im}( (\lambda_B)_I + (\varepsilon_K)_I).$$

Then $L_I = (H \oplus A)_I \oplus (K \oplus B)_I$ by Lemma 5.8. It follows that $1_H \cdot L_I = (H \oplus A)_I$, $1_A \cdot L_I = O$, $1_B \cdot L_I = 1_K \cdot L_I = (K \oplus B)_I$. Therefore, $F(L_I) = \left( \begin{array}{cc} H_I \\ O \\ 0 \\ (K \oplus B)_I \end{array} \right)$.

We show that $(H \oplus A)_I \cong H_I$ and $(K \oplus B)_I \cong K_I$ as left $H$-modules and left $K$-modules, respectively. The canonical projection $p_H : H \oplus A \to H$ induces a left $H$-module map $\overline{p_H} := p_H |(H \oplus A)_I : (H \oplus A)_I \to H_I$. Define a map $q : H_I \to (H \oplus A)_I$ by

$q(z) = ((\varepsilon_H)_I + (\lambda_A)_I)(z)$

for all $z \in H_I$. It can be easily verified that $\overline{p_H}$ and $q$ are inverse each other. Thus, we have $F(L_I) \cong \left( \begin{array}{cc} H_I \\ O \\ K_I \end{array} \right) = I$. This isomorphism, denoted by $\omega_F : I \to F(L_I)$, is given by $\omega_F = \left( \begin{array}{cc} q_H & 0 \\ 0 & q_K \end{array} \right)$, where

$q_H : H_I \to (H \oplus A)_I$, $q_H(z) = ((\varepsilon_H)_I + (\lambda_A)_I)(z),

q_K : K_I \to (K \oplus B)_I$, $q_K(w) = ((\varepsilon_K)_I + (\lambda_B)_I)(w)$

for all $z \in H_I$ and $w \in K_I$. One can show that the triple $(F, \text{Id}, \omega_F) : L \mathcal{M} \to \left( \begin{array}{cc} H_{B, M}^M A_{M}^M \\ B_{M}^M K_{M}^M \end{array} \right)$ is a monoidal functor, which gives an equivalence of $k$-linear monoidal categories. \qed
PROBLEM 5.10. Find the condition for the weak bialgebra $L$ in Theorem 5.9 such that $L$ is indecomposable.

We demonstrate a few examples based on Proposition 5.5.

LEMMA 5.11. Let us consider the 1-dimensional Hopf algebra $H = k$. We regard $A := k$ as an algebra by the product of $k$.

1. There is a unique weak $(k, k)$-comodule algebra structure for $A$, and it is given by $(A, \id_k, \id_k)$ under the identification $k \otimes k = k$.

2. Let $B := k$ be the same algebra with $A$, and $(A, \lambda_A, \rho_A), (B, \lambda_B, \rho_B)$ be two weak $(k, k)$-comodule algebras. Then a $k$-linear map $\Delta_{A,B} : k = H \rightarrow A \otimes B = k \otimes k \cong k$ preserves products and satisfies the conditions (1), (2), (4), (5), (6) in Proposition 5.5 if and only if $(\lambda_A, \rho_A, \lambda_B, \rho_B, \Delta_{A,B}) = (\id, \id, \id, \id, \id)$.

Proof. (1) Let $A = (A, \lambda_A, \rho_A)$ be a weak $(k, k)$-comodule algebra. By the definition of left comodule, $(\varepsilon \otimes \id_A) \circ \lambda_A = \id_A$ holds. This implies that $\lambda_A \neq 0$. Similarly, $\rho_A \neq 0$. We view $\lambda_A$ as the map $\lambda_A : k = A \rightarrow H \otimes A = k \otimes k \cong k$. Since $\lambda_A(1) = \lambda_A(1)\lambda_A(1)$, we have $\lambda_A(1) = 0$ or $\lambda_A(1) = 1$.

If $\lambda_A(1) = 0$, then $\lambda_A(a) = \lambda_A(a \cdot 1) = \lambda_A(a)\lambda_A(1) = 0$ for all $a \in k$. This implies that $\lambda_A = 0$, that is a contradiction. Thus $\lambda_A(1) = 1$. By linearity of $\lambda_A$, in this case $\lambda_A(a) = \lambda_A(a \cdot 1) = a\lambda_A(1) = a \cdot 1 = a$ for all $a \in A = k$. It follows that $\lambda_A = \id_k$. The condition (LWCA2) is satisfied since it can be expressed as $a = \varepsilon(\lambda_A(a)) = \lambda_A(a)$. The condition (LWCA1) is also satisfied.

Similarly, viewing $\rho_A$ as $\rho_A : k = A \rightarrow A \otimes H = k \otimes k \cong k$, we see that $\rho_A = \id_k$.

Conversely, when $\lambda_A = \rho_A = \id$, it can be immediately shown that $(A, \id, \id)$ is a weak $(k, k)$-comodule algebra.

(2) Since $\Delta_{A,B}$ preserves products, it can be shown that $\Delta_{A,B} = 0$ or $\Delta_{A,B} = \id$ by a similar consideration of the proof of (1). If $\Delta_{A,B} = 0$, then the conditions (1), (2), (4), (5) in Proposition 5.5 are satisfied. The condition (6) in Proposition 5.5 is satisfied if and only if $\rho_A = 0$ or $\lambda_B = 0$. However, it does not occur by (1). So, $\Delta_{A,B} = \id$ is required. The condition (1) in Proposition 5.5 requires $\rho_B = \id$ and $\lambda_A = \id$. When $\rho_B = \id$ and $\lambda_A = \id$, the conditions (4) and (5) are satisfied. Similarly, the condition (6) in Proposition 5.5 requires $\rho_A = \id$ and $\lambda_B = \id$.

EXAMPLE 5.12. Let $H = K = k$ and set $A = k, B = 0$. Then $\Delta_{A,B} = 0$ and $\Delta_{B,A} = 0$. It follows that there is a unique 3-dimensional weak bialgebra obtained by the construction in Proposition 5.5, which is given in the form 

\[
\begin{pmatrix}
  k & k \\
  0 & k
\end{pmatrix}
\]

This weak bialgebra is isomorphic to $L := ke_1 \oplus ke_2 \oplus ke_3$ whose algebra and coalgebra structures are given by

\[
e_i e_j = \delta_{ij} e_i \quad \text{for all } i, j = 1, 2, 3,
\]

\[
\Delta_L(xe_1 + ae_2 + ye_3) = xe_1 \otimes e_1 + a(e_2 \otimes e_3 + e_1 \otimes e_2) + ye_3 \otimes e_3,
\]

\[
\varepsilon_L(xe_1 + ae_2 + ye_3) = x + y
\]

for all $x, a, y \in k$. The weak bialgebra $L$ is indecomposable as a weak bialgebra, and isomorphic to the 11th 3-dimensional weak bialgebra in the list of [7, Proposition 4.5]. This can be verified by using the basis $e_1 + e_2 + e_3, e_1 + e_3, e_1$. Thus, $L$ is
isomorphic to the dual of the weak bialgebra spanned by the morphisms of the interval category $2$. (See Example 3.7.)

Example 5.13. Let us consider the case where $H = K = k$ and $A = B = k$ in Proposition 5.5. Let $A = (A, \lambda_A, \rho_A), \ B = (B, \lambda_B, \rho_B)$ be a weak $(H, K)$-comodule algebra and a weak $(K, H)$-comodule algebra, respectively. By Lemma 5.11(2), there is a unique $(\lambda_A, \rho_A, \lambda_B, \rho_B, \Delta_A, B, \Delta_B, A)$ such that it satisfies all conditions in Proposition 5.5, and it is given by $(\text{id}, \text{id}, \text{id}, \text{id}, \text{id}, \text{id})$.

The algebra structure of the weak bialgebra $L = H \otimes A \otimes B \otimes K$ defined as in the proposition is the direct product of four $k$s. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $L$, then $e_i e_j = \delta_{ij} e_i (i, j = 1, 2, 3, 4)$ and the coalgebra structure of $L$ is given as follows:

$$\Delta_L(x_1 + a e_2 + b e_3 + y e_4) = x(e_1 \otimes e_1 + e_2 \otimes e_3) + a(e_1 \otimes e_2 + e_2 \otimes e_4) + b(e_4 \otimes e_3 + e_3 \otimes e_1) + y(e_4 \otimes e_4 + e_3 \otimes e_2),$$

$$\varepsilon_L(x_1 + a e_2 + b e_3 + y e_4) = x + y$$

for all $x, a, b, y \in k$. Since $\varepsilon_L(1_L) = 2 \neq 1$, it follows that the weak bialgebra $L$ is not a bialgebra. Furthermore, $L$ has an antipode $S_L$, which is given by $S_L(x_1 + a e_2 + b e_3 + y e_4) = x e_1 + y e_4$ for all $x, a, b, y \in k$. The weak Hopf algebra $L$ is indecomposable.

Remark 5.14. The above $L$ can not be constructed from a groupoid. Especially, it is not isomorphic to the qubit groupoid algebra since $G(L) = \{ e_1 + a e_2 + a^{-1} e_3 + e_4 \mid a \in k^4 \} \cong k^4$ and the order of $G(L)$ does not coincide with $\dim L = 4$. In fact, $L$ is isomorphic to the dual of the qubit groupoid algebra.

Acknowledgments. I would like to thank Professor Akira Masuoka for discussing and suggesting on the direct sum construction, and thank Professor Kenichi Shimizu for suggesting and explaining on interpretation of indecomposability by a categorical language and a generalization of the direct sum construction. I would like to thank the organizers, Professors Nicolás Andruskiewitsch, Gongxiang Liu, Susan Montgomery, Yinhuo Zhang, for inviting me to the International Workshop on Hopf algebras and Tensor Categories at Nanjing in China on September 9–13, 2019. Much thanks are also graduate students and young researchers in Nanjing University for kind hospitality during my stay in China. Helpful comments from Professors Nicolás Andruskiewitsch and Gabriella Böhm in the workshop provide me with inspirations for construction of examples of indecomposable weak Hopf algebras in this paper. Finally, I would like to thank the referees for careful reading and for many helpful comments and suggestions to improve the previous manuscript.

References


Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita-shi, Osaka 564-8680, Japan
E-mail address: wakui@kansai-u.ac.jp