Equivalence Problem of Geometric Structure

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On the auspicious occasion of
Prof. Miyaoka and Prof. Yamaguchi
brimming with Math spirit

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Abstruct

There has been many results on equivalence problems of geometric structures along with the method of É. Cartan, which consists of several processes of prolongations to higher order jet space, absorptions of torsion, extracting curvatures, and so on. In a series of joint works with Prof. H. Sato, we introduced *systems of linear PDEs* in connection with certain geometric structures, such that the integrability condition of the system is equal to the vanishing of curvatures and its solutions give the equivalence maps.

In this talk, I will discuss on pseudo-Hermitian structure (roughly speaking, a contact form together with an integrable complex structure J along the contact distribution), and give a system of linear PDEs that solves the equivalence problem for pseudo-Hermitian structure in dimension three.

Contact structure on Heisenberg group

$$H = \mathbb{C} \oplus \mathbb{R} \ni (z,t) \ ; \ (z,t)\cdot(w,s) = (z+w, t+s-2\Im(z\bar{w}))$$

is the *Heisenberg group*. Using notation $\mathbb{C} \oplus \mathbb{R} = \mathbb{R}^3 \ni (z,t) = (x,y,t)$, define right invariant vector fields

$$v_1 = \frac{1}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad v_2 = \frac{1}{2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \quad v_3 = \frac{\partial}{\partial t}$$

and 1-forms

$$\alpha_1 = 2dx, \quad \alpha_2 = 2dy, \quad \alpha_3 = dt + 2(xdy - ydx) = dt + \sqrt{-1}(zd\bar{z} - \bar{z}dz)$$

are called *Heisenberg frame* and *Heisenberg coframe*, which satisfy

$$[v_2, v_1] = v_3, \quad [v_3, v_1] = [v_3, v_2] = 0,$$
$$d\alpha_3 = \alpha_1 \wedge \alpha_2, \quad d\alpha_1 = d\alpha_2 = 0.$$

Fix a contact form $e^{\eta}\alpha_3$, and denote by $\mathcal{D}$ the contact distribution, which is spaned by v_1 and v_2;

$$\mathcal{D} = \{v_1 \, , \, v_2\}.$$

The *Reeb vector field* of the contact form $e^{\eta}\alpha_3$) is

$$T_\eta = e^{-\eta}v_2(\eta)v_1 - e^{-\eta}v_1(\eta)v_2 + e^{-\eta}v_3.$$

and satisfying $e^{\eta}\alpha_3(T_\eta) = 1$ and $\mathcal{L}_{T_\eta}(e^{\eta}\alpha_3) = 0$.

CR structure based on $\mathcal{D}$

A *CR structure* on $(H, \mathcal{D})$ is a decomposition

$$\mathbb{C} \otimes \mathcal{D} = \mathcal{D}_{(1,0)} \oplus \mathcal{D}_{(0,1)}.$$

For any complex valued functions f and g on H, the line fields

$$\mathcal{D}_{(1,0)} = \mathbb{C}(fv_1 + gv_2) \text{ and } \mathcal{D}_{(0,1)} = \mathbb{C}(\bar{f}v_1 + \bar{g}v_2)$$

defines a CR structure if and only if the imaginary part of $\sigma = g/f$ does not vanish;

$$\Im(\sigma) = \Im(g/f) \neq 0.$$

Let $Z_\sigma = v_1 + \sigma v_2$.

$$\begin{array}{ccc} \{\text{CR structures on } (H,\mathcal{D})\} & \longleftrightarrow & \{\sigma : H \to \mathbb{C} \ ; \ \Im(\sigma) \neq 0\} \\ \Cup & & \Cup \\ \mathbb{C}Z_\sigma \oplus \mathbb{C}\bar{Z}_\sigma & \longleftrightarrow & \sigma \end{array}$$

Integrability condition is automatic, since H is 3-dimensional.

Given a CR structure Z_σ, a unique linear complex structure J_σ of each $\mathcal{D}_p$ whose complexification has the same eigenspace decomposition $\mathbb{C}Z_\sigma \oplus \mathbb{C}\bar{Z}_\sigma$.

$$Z_\sigma \quad \longleftrightarrow \quad \sigma \quad \longleftrightarrow \quad J_\sigma$$

Equivalence of pseudo-Hermitian structure

Definition. A *pseudo-Hermitian structure* on U is a combination

$$\left\{ \begin{array}{rcl} \text{a contact form} & : & e^\eta \alpha_3 \\ \text{a CR structure} & : & Z_\sigma = v_1 + \sigma v_2 \end{array} \right\} \quad = \quad (U, \sigma, \eta).$$

A diffeomorphism $\Phi : U_1 \to U_2$ between two pseudo-Hermitian manifolds (U_i, σ_i, η_i) $(i = 1, 2)$ is an equivalence map if and only if it satisfies

$$(1)\ \Phi_* : TU_1 \to TU_2 \text{ commutes with } J_{\sigma_i}, \quad \text{and} \quad (2)\ \Phi^*(e^{\eta_2}\alpha_3) = e^{\eta_1}\alpha_3$$

which is necessarily a contact map.

Regard $(V, \sigma = -\sqrt{-1}, \eta = 0)$ as a standard model $(V \subset H)$.

PROBLEM

Under what conditions does there exist an equivalence map

$$\Phi : (U, \sigma, \eta) \to (V, -\sqrt{-1}, 0)$$

for some $V \subset H$? If this is the case, how can one find Φ ?

Main Theorem

Let (U, σ, η) be a pseudo-Hermitian structure.

$$\left\{ \begin{array}{l} 0 = \bar{Z}_\sigma(f) \\ 0 = {Z_\sigma}^2(f) - (v_2(\sigma) + 2Z_\sigma(\eta))Z_\sigma(f) \end{array} \right. \tag{F}$$

The maximal dimension of the solution space of (F) is 3; an initial condition

$$(f(p), Z_\sigma(f)(p), \bar{Z}_\sigma Z_\sigma(f)(p)) \in \mathbb{C}^3.$$

Theorem 1. (F) is integrable $\iff$ $\tau = \kappa = 0$

Theorem 2. *Suppose (U, σ, η) satisfies $\tau = \kappa = 0$. Then for an orth-normal basis $\{f_1, f_2, f_3\}$ with $f_1 = 1/2$ of the solution space of (F),*

$$\Phi := (f_2, -\Im(f_3)) : (U, \sigma, \eta) \to (H, -\sqrt{-1}, 0)$$

is an equivalence map, where $\Im$ is the imaginary part.

♡ **Hermitian product on the solution space of (F)**

$$\langle f, g\rangle_{(\sigma,\eta)} = -\sqrt{-1}\big(fT_\eta(\bar{g}) - T_\eta(f)\bar{g}\big) + h^{-1} Z_\sigma(f)\bar{Z}_\sigma(\bar{g})$$

where h is the coefficient of the *Levi form* with respect to $\{Z_\sigma, \bar{Z}_\sigma\}$;

$$h := L(Z_\sigma, Z_\sigma) = -\sqrt{-1}d(e^\eta\alpha_3)(Z_\sigma, \bar{Z}_\sigma) = \sqrt{-1}(\sigma - \bar{\sigma})e^\eta.$$

♡ **p-H torsion τ and T-W curvature κ of (U, σ, η)**

$$\begin{aligned}
\tau = \frac{-e^{-\eta}}{\sigma - \bar{\sigma}}\Big(& v_2(\eta)v_1(\bar{\sigma}) - v_1(\eta)v_2(\bar{\sigma}) - \big(v_1(\eta) + \bar{\sigma}v_2(\eta)\big)^2 \\
& + v_1v_1(\eta) + \bar{\sigma}^2 v_2v_2(\eta) + v_3(\bar{\sigma}) + \bar{\sigma}v_4(\eta)\Big) \\
\kappa = & -2(\bar{\sigma}\, v_2(\sigma) - \sigma v_2(\bar{\sigma}))^2/(\sigma - \bar{\sigma})^2 \\
& + \Big(4\left(\sigma\, v_2(\bar{\sigma}) - \bar{\sigma}\, v_2(\sigma)\right) v_1(\eta) + 4\left(\sigma^2\, v_2(\bar{\sigma}) - \bar{\sigma}^2\, v_2(\sigma)\right) v_2(\eta) \\
& \qquad - v_1(\bar{\sigma})v_2(\sigma) + v_1(\sigma)v_2(\bar{\sigma}) - v_1v_1(\sigma - \bar{\sigma}) \\
& \qquad + \sigma^2\, v_2v_2(\bar{\sigma}) - \bar{\sigma}^2\, v_2v_2(\sigma) - \bar{\sigma}v_4(\sigma) + \sigma v_4(\bar{\sigma})\Big)\Big/(\sigma - \bar{\sigma}) \\
& - \Big(2\,(v_1v_1 + \sigma\bar{\sigma}v_2v_2)(\eta) + (\sigma + \bar{\sigma})\, v_4(\eta) \\
& \qquad + v_1(\eta)^2 + \sigma\,\bar{\sigma}\, v_2(\eta)^2 + (\sigma + \bar{\sigma})v_1(\eta)\, v_2(\eta) + v_1(\sigma + \bar{\sigma})\, v_2(\eta)\Big)
\end{aligned}$$

where $v_4 = v_1v_2 + v_2v_1$.

Standard model

Real hypersurfaces. For a real hypersurface M in the 2-dimensional complex space ($\mathbb{C}^2$ or $\mathbb{C}P^2$), the intersection

$$\mathcal{D}_p = T_pM \cap \sqrt{-1}T_pM$$

is always 2-dimensional, which gives a contact structure on M.

$\mathcal{D}_p$ is closed under the multiplication $J := \times\sqrt{-1}$, and J defines a CR structure on M.

Our model $$\boxed{(H, \sigma = -\sqrt{-1}, \eta = 0)}$$

$$\begin{array}{ccccccccc} H & \ni & (z,t) & \mapsto & (z, |z|^2 - \sqrt{-1}t) & \in & \{\Re(w) = |z|^2\} & \subset & \mathbb{C}^2_{(z,w)} \\ & & \searrow & & \downarrow & & \downarrow & & \\ & & \varphi_s & & [\frac{1}{2}, z, |z|^2 - \sqrt{-1}t] & \in & \{z_0\bar{z}_2 + z_2\bar{z}_0 = |z_1|^2\} & \subset & \mathbb{C}P^2_{[z_0,z_1,z_2]} \end{array}$$

$e^0\alpha_3 = \alpha_3 = dt + \sqrt{-1}(zd\bar{z} - \bar{z}dz)$ is the standard contact form on H.

The map $\varphi : H \to \mathbb{C}P^2$ consists of functions $\frac{1}{2}$, z, $z\bar{z} - \sqrt{-1}t$, which form an orthnormal basis of the solution space of equation

$$\begin{cases} 0 = \bar{Z}_{-\sqrt{-1}}(f) & = (v_1 + \sqrt{-1}v_2)(f) \\ 0 = (Z_{-\sqrt{-1}})^2(f) & = (v_1 - \sqrt{-1}v_2)^2(f) \end{cases}$$

Remark. If $\eta = 0$(i.e. α_3 were chosen as the contact form), then the Reeb field is $T_{\eta=0} = v_3$.

Remark. If $\sigma \equiv -\sqrt{-1}$, then $J_{-\sqrt{-1}}$ maps v_1 and v_2 to

$$J_{-\sqrt{-1}}(v_1) = v_2 \quad \text{and} \quad J_{-\sqrt{-1}}(v_2) = -v_1.$$

Introduce a Hermitian structure $\langle\ ,\ \rangle_s$

$$\begin{aligned} &\langle f, g\rangle_s \\ &= \bar{Z}_{-\sqrt{-1}}Z_{-\sqrt{-1}}(f)\cdot\bar{g} + f\cdot Z_{-\sqrt{-1}}\bar{Z}_{-\sqrt{-1}}(\bar{g}) - Z_{-\sqrt{-1}}(f)\cdot\bar{Z}_{-\sqrt{-1}}(\bar{g}). \end{aligned}$$

Then the 3 functions $f_0 = \frac{1}{2}$, $f_1 = z$, $f_2 = z\bar{z} - \sqrt{-1}t$ satisfy

$$\Big(\langle f_i, f_j\rangle_s\Big)_{i,j} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \leftrightarrow \quad z_0\bar{z}_2 + z_2\bar{z}_0 - |z_1|^2 = 0$$

Remark. A 4-dimensional Lie group acts on $\mathbb{C}P^2$ preserving $(S^3, \varphi_*(\alpha_3))$ invariant.

Idea to solve the problem

$$\begin{array}{ccc} (H, -\sqrt{-1}, 0) & \xrightarrow{\varphi_s} & \mathbb{C}P^2 \\ \Phi \uparrow & \nearrow_{\varphi} & \\ (U, \sigma, \eta) & & \end{array}$$

On the standard model $(H, -\sqrt{-1}, 0)$, we have

$$(\mathrm{F_s}) \quad \begin{cases} 0 = \bar{Z}_{-\sqrt{-1}}(f) \\ 0 = (Z_{-\sqrt{-1}})^2(f) \end{cases} \qquad \text{and} \qquad \langle\ ,\ \rangle_{\mathrm{s}} \text{ on } \mathrm{Sol}(\mathrm{F_s})$$

$(\mathrm{s} = (\sigma = -\sqrt{-1}, \eta = 0))$

Imagine we have a contact diffeomorphism $\Phi : U \to H$, and pull back the standard CR structure and the standard contact form;

$$e^{\xi} Z_{\sigma} = e^{\xi}(v_1 + \sigma v_2) := \Phi^*(Z_{-\sqrt{-1}}), \quad e^{\eta}\alpha_3 := \Phi^*(\alpha_3)$$

getting ξ, σ, η.

$$\Phi^*(\bar{Z}_{-\sqrt{-1}}) = e^{\bar{\xi}} \bar{Z}_{\sigma}, \quad \Phi^*((Z_{-\sqrt{-1}})^2) = e^{2\xi}\left((Z_{\sigma})^2 + Z_{\sigma}(\xi) Z_{\sigma}\right)$$

The funcamental equation

$$(\mathrm{F}_{(\sigma,\eta,\xi)}) \quad \begin{cases} 0 = \bar{Z}_{\sigma}(f) \\ 0 = \left((Z_{\sigma})^2 + Z_{\sigma}(\xi) Z_{\sigma}\right)^2(f) \end{cases}$$

and the inner product $\langle\ ,\ \rangle_{(\sigma,\eta,\xi)}$ on $\mathrm{Sol}(\mathrm{F}_{(\sigma,\eta,\xi)})$

$$\begin{aligned} \langle f, g\rangle_{(\sigma,\eta,\xi)} = & e^{\xi+\bar{\xi}}\Big(\bar{Z}_{\sigma} Z_{\sigma}(f) \cdot \bar{g} + f \cdot Z_{\sigma}\bar{Z}_{\sigma}(\bar{g}) - Z_{\sigma}(f) \cdot \bar{Z}_{\sigma}(\bar{g}) \\ & + \bar{Z}_{\sigma}(\xi) Z_{\sigma}(f) \cdot g + f \cdot Z_{\sigma}(\bar{\xi}) \bar{Z}_{\sigma}(g)\Big) \end{aligned}$$

$$\begin{array}{ccc} & \Phi & \\ \swarrow & & \searrow \\ (\sigma, \eta) & \overset{?}{\dashrightarrow} & \underbrace{\Phi^*(\mathrm{F_s}), \Phi^*(\langle\ ,\ \rangle_{\mathrm{s}})}_{\xi, \sigma, \eta} \end{array}$$

Key Lemma

Lemma. *Let φ be a contact diffeomorphism. Put $\varphi(\alpha_3) = e^{\eta}\alpha_3$, and $\varphi^*(Z_s) = e^{\xi} Z_\sigma$. Then it holds that*

$$Z_\sigma(\xi) = -2Z_\sigma(\eta) - v_2(\sigma) \quad \text{and} \quad Z_\sigma(\bar{\xi}) = Z_\sigma(\eta) - \frac{Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma)}{\bar{\sigma} - \sigma}.$$

Proof. Let $e^{\eta}\alpha_3 = \varphi^*(\alpha_3)$ be the pullback of the contact form α_3. The vector fields v_3 and

$$e^{-\eta}v_2(\eta)v_1 - e^{-\eta}v_1(\eta)v_2 + e^{-\eta}v_3$$

are the Reeb vector fields with respect to the contact forms α_3 and $e^{\eta}\alpha_3$, respectively. Therefore, one obtains

$$e^{-\eta}v_2(\eta)v_1 - e^{-\eta}v_1(\eta)v_2 + e^{-\eta}v_3 = \varphi^*(v_3). \tag{1}$$

Since $\varphi^*(Z_s) = e^{\xi} Z_\sigma$, one has

$$\varphi^*([Z_s, \bar{Z}_s]) = [e^{\xi} Z_\sigma, e^{\bar{\xi}} \bar{Z}_\sigma]. \tag{2}$$

An easy calculation shows that

$$[Z_s, \bar{Z}_s] = -2\sqrt{-1}v_3 \tag{3}$$

and

$$\begin{aligned} [e^{\xi} Z_\sigma, e^{\bar{\xi}} \bar{Z}_\sigma] \;&=\; e^{\xi+\bar{\xi}} \left(Z_\sigma(\bar{\xi}) - \bar{Z}_\sigma(\xi)\right) v_1 \\ &\quad + e^{\xi+\bar{\xi}} \left(\bar{\sigma} Z_\sigma(\bar{\xi}) - \sigma \bar{Z}_\sigma(\xi) + (Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma))\right) v_2 \\ &\quad + e^{\xi+\bar{\xi}}(\sigma - \bar{\sigma}) v_3 \end{aligned} \tag{4}$$

By (??), (??), (??) and (??), we get

$$-2\sqrt{-1}e^{-\eta}v_2(\eta) = e^{\xi+\bar{\xi}} \left(Z_\sigma(\bar{\xi}) - \bar{Z}_\sigma(\xi)\right) \tag{5}$$

$$2\sqrt{-1}e^{-\eta}v_1(\eta) = e^{\xi+\bar{\xi}} \left(\bar{\sigma} Z_\sigma(\bar{\xi}) - \sigma \bar{Z}_\sigma(\xi) + (Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma))\right) \tag{6}$$

$$-2\sqrt{-1}e^{-\eta} = e^{\xi+\bar{\xi}}(\sigma - \bar{\sigma}). \tag{7}$$

By (??) and (??), we get

$$2\sqrt{-1}e^{-\eta} Z_\sigma(\eta) = e^{\xi+\bar{\xi}} \left((\bar{\sigma} - \sigma) Z_\sigma(\bar{\xi}) + (Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma))\right),$$

and thus by (??) we obtain

$$Z_\sigma(\bar{\xi}) = Z_\sigma(\eta) - \frac{Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma)}{\bar{\sigma} - \sigma}. \tag{8}$$

Derivating (??) by Z_σ, we obtain

$$2\sqrt{-1}e^{-\eta} Z_\sigma(\eta) = e^{\xi+\bar{\xi}} \left((\sigma - \bar{\sigma}) Z_\sigma(\xi + \bar{\xi}) + Z_\sigma(\sigma - \bar{\sigma})\right),$$

and thus by (??) we obtain

$$-Z_\sigma(\eta) = Z_\sigma(\xi) + Z_\sigma(\bar{\xi}) - \frac{Z_\sigma(\sigma - \bar{\sigma})}{\sigma - \bar{\sigma}}. \tag{9}$$

From (??) and (??), it follows

$$Z_\sigma(\xi) = -2Z_\sigma(\eta) - v_2(\sigma).$$

q.e.d.

The frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ satisfies

$$\begin{cases} [Z_\sigma, \bar{Z}_\sigma] = -mZ_\sigma + \bar{m}\bar{Z}_\sigma - \sqrt{-1}hT \\ [Z_\sigma, T] = pZ_\sigma - q\bar{Z}_\sigma, \\ [\bar{Z}_\sigma, T] = -\bar{q}Z_\sigma + \bar{p}\bar{Z}_\sigma \end{cases}$$

where $q = -\bar{\tau}$ holds.

Suppose $\tau = \kappa = 0$. Then we have

$$[Z_\sigma, T] = pZ_\sigma,$$

and for a solution f of $(F_{(\sigma,\eta)})$ we have equalities

$$\begin{aligned} &\bar{Z}_\sigma Z_\sigma(f) = mZ_\sigma(f) + \sqrt{-1}hT(f), \\ &Z_\sigma T(f) = \bar{Z}_\sigma T(f) = TZ_\sigma(f) + pZ_\sigma(f) = T^2(f) = 0 \end{aligned}$$

The torsion and the curvature are

$$\begin{aligned} \tau &= \frac{-1}{\sigma - \bar{\sigma}}\Big(T(\bar{\sigma}) - \bar{\sigma}\bar{Z}_\sigma\big(v_2(e^{-\eta})\big) - \bar{Z}_\sigma\big(v_1(e^{-\eta})\big)\Big) \\ \kappa &= -\bar{Z}_\sigma(s) + Z_\sigma(m) + m(s - \bar{m}) - \sqrt{-1}hp. \end{aligned}$$

where $s = v_2(\sigma) + 2Z_\sigma(\eta)$.

How to prove the integrability of (F)

Lemma. *Suppose (U, σ, η) is a pseudo-Hermitian structure with $\tau = \kappa = 0$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be matrices defined by*

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & \sqrt{-1}h \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$(s = 2Z_\sigma(\eta) + v_2(\sigma))$ and let $\tilde{f}$ denote the column vector $\big(f \ Z_\sigma(f) \ T(f)\big)^t$. Then the system of equations (F) is equivalent to the following system of equations:

$$Z_\sigma(\tilde{f}) = \mathcal{A}\tilde{f}, \quad \bar{Z}_\sigma(\tilde{f}) = \mathcal{B}\tilde{f}, \quad T(\tilde{f}) = \mathcal{C}\tilde{f}$$

Vector fields:

$$[v_i, v_j] = \sum_{k=1}^{n} \gamma_{ij}^k v_k.$$

System of linear PDE:

$$v_i(f) = S_i f \quad \text{for } i = 1, \cdots, n,$$

Zero curvature condition:

$$v_i(S_j) - v_j(S_i) + [S_j, S_i] = \sum_{k=1}^{n} \gamma_{ij}^k S_k \quad \text{for all } i, j = 1, \cdots, n.$$

Transformation of the fundamental equation

Our fundamental equation (F) is

$$\begin{cases} 0 = \bar{Z}_\sigma(f) \\ 0 = {Z_\sigma}^2(f) - (v_2(\sigma) + 2Z_\sigma(\eta))Z_\sigma(f) \end{cases} \tag{F}$$

Below, we use the notation σ_i to mean $v_i(\sigma)$. Reminding $Z_\sigma = v_1 + \sigma v_2$, and derivating the first equation by v_1 and v_2, we get

$$\begin{aligned} {v_1}^2(f) + \bar{\sigma} v_1 v_2(f) &= -\bar{\sigma}_1 v_2(f) \\ v_2 v_1(f) + \bar{\sigma} {v_2}^2(f) &= -\bar{\sigma}_2 v_2(f) \\ {v_1}^2(f) + \sigma(v_1 v_2(f) + v_2 v_1(f)) + \sigma^2 {v_2}^2(f) &= A v_1(f) + B v_2(f) \\ v_1 v_2(f) - v_2 v_1(f) &= -v_3(f), \end{aligned}$$

where $A = \sigma_2 + 2(\eta_1 + \sigma\eta_2)$, and $B = -\sigma_1 + 2\sigma(\eta_1 + \sigma\eta_2)$. Therefore we get

$$v_i v_j(f) = \sum_{k=1}^{2} \Gamma_{ij}^k v_k(f) + g_{ij} v_3(f) \qquad (i, j = 1, 2),$$

where

$$g_{11} = -\frac{\sigma\bar{\sigma}}{\sigma - \bar{\sigma}}, \quad g_{12} = \frac{\bar{\sigma}}{\sigma - \bar{\sigma}}, \quad g_{21} = \frac{\sigma}{\sigma - \bar{\sigma}}, \quad g_{22} = -\frac{1}{\sigma - \bar{\sigma}}$$

and

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix} = \frac{1}{(\sigma - \bar{\sigma})^2} \begin{pmatrix} \bar{\sigma}^2 A & \bar{\sigma}^2 B - \sigma(\sigma - 2\bar{\sigma})\bar{\sigma}_1 + \sigma^2\bar{\sigma}\bar{\sigma}_2 \\ -\bar{\sigma} A & -\bar{\sigma} B - \bar{\sigma}\bar{\sigma}_1 - \sigma^2\bar{\sigma}_2 \\ -\bar{\sigma} A & -\bar{\sigma} B - \bar{\sigma}\bar{\sigma}_1 - \sigma^2\bar{\sigma}_2 \\ A & B + \bar{\sigma}_1 - (\bar{\sigma} - 2\sigma)\bar{\sigma}_2 \end{pmatrix}.$$

$(A = \sigma_2 + 2(\eta_1 + \sigma\eta_2)$, $B = -\sigma_1 + 2\sigma(\eta_1 + \sigma\eta_2))$

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