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Bäcklund Transformations for Darboux Integrable Equations

Differential Geometry and Tanaka Theory -RIMS

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January 24, 2011

Abstract

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Abstract

Lie symmetry reduction is typically viewed as an integration method for differential systems of finite type, that is, systems of ordinary differential equations.

In this talk I shall present two new, recent applications of Lie symmetry reduction to the study of partial differential equations.

The first gives a remarkably simple method for constructing Bäcklund transformations.

The second also gives a simple, very general method for constructing Darboux integrable equations.

The combination of these result in a new method for constructing Bäcklund transformations for Darboux integrable equations.

The utility of this group theoretic approach will be illustrated by a variety of novel examples.

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Example

The Bäcklund transformation between $z_{xy} = 0$ and the Liouville equation $u_{xy} = \exp(u)$ is the first order system of PDE:

$$z_x + u_x = -\sqrt{2}\exp\frac{z-u}{2}, \quad z_y - u_y = \sqrt{2}\exp\frac{z+u}{2}$$

If we treat u as given, then:

$$z_x = -u_x - \sqrt{2}\exp\frac{z-u}{2}, \quad z_y = u_y + \sqrt{2}\exp\frac{z+u}{2}$$

If we treat z as given, then:

$$u_x = -z_x - \sqrt{2}\exp\frac{z-u}{2}, \quad u_y = z_y - \sqrt{2}\exp\frac{z+u}{2}$$

To formalize the notion of a Bäcklund transformation within the context of differential systems theory, we recall the definition of an *integrable extension.*

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Integrable Extensions

Let $\mathcal{I}$ be an EDS on M.

An EDS $\mathcal{E}$ on N is an integral extension of $\mathcal{I}$ if:

$$\mathbf{p} : N \to M \quad \text{and a Pfaffian system } J \text{ on } N$$

with

$$\mathcal{E} = \langle \mathbf{p}^*(\mathcal{I}) + \mathcal{S}(J) \rangle_{\mathrm{alg}}.$$

We also require that

$$\operatorname{rank} J = \dim N - \dim M,$$

and that J is transverse:

$$\operatorname{ann}(J) \cap \ker (\mathbf{p}_*) = 0\,,$$

If $\mathcal{I} =< \theta^i >_{\mathrm{alg}}$, then $\mathcal{E} =< \mathbf{p}^*(\theta^i), \zeta^a >_{\mathrm{alg}}$ and

$$d\zeta^a \equiv 0 \mod \{ \mathbf{p}^*(\theta^i), \zeta^a \}$$

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Bäcklund Transformations

Differential systems $\mathcal{I}_1$ and $\mathcal{I}_2$ are related by a ***Bäcklund transformation*** if there exists a system $\mathcal{B}$ which is ***simultaneously*** an integrable extension for both $\mathcal{I}_1$ and $\mathcal{I}_2$

$$\begin{array}{ccc} & (\mathcal{B}, N) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ (\mathcal{I}_1, M_1) & & (\mathcal{I}_2, M_2) \end{array} \quad .$$

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Example Part 2

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$$z_x + u_x = -\sqrt{2}\exp\frac{z-u}{2}, \quad z_y - u_y = \sqrt{2}\exp\frac{z+u}{2},$$

WAVE: $z_{xy} = 0$

$$M_1 = (x, y, z, z_x, z_y)$$

$$\mathcal{I}_1 = \langle\ dz - z_x dx - z_y dy,\ dz_x \wedge dx,\ dz_y \wedge dy\ \rangle_{alg}.$$

BÄCKLUND - VERSION I

$$M_{\text{ver I}} = (x, y, z, z_x, z_y, u)$$

$$\mathcal{B}_{\text{ver I}} = \langle \mathcal{I}_1,\ \zeta_I \rangle_{\text{alg}}$$

$$\zeta_I = du + (z_x + \sqrt{2}e^{\frac{-z+u}{2}})\,dx - (z_y - \sqrt{2}e^{\frac{z+u}{2}})\,dy,$$

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$$z_x + u_x = -\sqrt{2}\exp\frac{z-u}{2}, \quad z_y - u_y = \sqrt{2}\exp\frac{z+u}{2},$$

Liouville:

$$N_2 = (x, y, u, u_x, u_y)$$

$$\mathcal{I}_2 = \langle\ du - u_x dx - u_y dy,\ (du_x - e^u dy)\wedge dx,\ (du_y - e^u dx)\wedge dy\rangle_{alg}.$$

Bäcklund - Version II

$$M_{\text{ver II}} = (x, y, u, u_x, u_y, z)$$

$$\mathcal{B}_{\text{ver II}} = \langle \mathcal{I}_2,\ \zeta_{\text{II}}\rangle_{\text{alg}}$$

$$\zeta_{\text{II}} = \ dz + (u_x + \sqrt{2}e^{\frac{-z+u}{2}})\,dx - (u_y + \sqrt{2}e^{\frac{z+u}{2}})dy,$$

We shall give a group theoretic derivation of this Bäcklund transformation

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Symmetry Reduction

Transformation groups are used in different ways to solve differential equations.

Group Invariant Solutions:

$$u_{xx} + u_{yy} = 0 \implies u = f(x^2 + y^2) \implies f'' + \frac{1}{r} f' = 0.$$

Symmetry Reduction:

$$y'' = 0 \implies z = y' \implies z' = 0.$$

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Symmetry Reduction:

$$y'' = 0 \implies z = y' \implies z' = 0.$$

Let G be a Lie group acting on M and set $\mu_g(x) = g \cdot x$. Then G is a symmetry group of $\mathcal{I}$ if

$$\mu_g^*(\mathcal{I}) = \mathcal{I} \quad \text{for all } g \in G$$

Let Γ be the Lie algebra of infinitesimal generators for the action of G. Then

$$\mathcal{L}_X \mathcal{I} \subset \mathcal{I} \quad \text{for all } X \in \Gamma$$

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From here on we suppose that M/G is a smooth manifold with

$$\mathbf{q}_G : M \to M/G \quad \text{a smooth submersion.}$$

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From here on we suppose that M/G is a smooth manifold with

$$\mathbf{q}_G \colon M \to M/G \quad \text{a smooth submersion.}$$

The G - reduction of $\mathcal{I}$ is:

$$\mathcal{I}/G = \{\ \theta \in \Omega^*(M/G) \mid \mathbf{q}_G^* \theta \in \mathcal{I}\ \}.$$

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$$\mathcal{I}/G = \{\ \theta \in \Omega^*(M/G) \mid \mathbf{q}_G^* \theta \in \mathcal{I}\ \}.$$

Under the assumption of transversality:

$$X \in \Gamma \text{ and } X \lrcorner\, \alpha = 0 \text{ for } \alpha \in \mathcal{I}^1 \Longrightarrow X = 0$$

explicit calculations of $\mathcal{I}/G$ are not difficult.

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Recognition Problem:

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explicit calculations of $\mathcal{I}/G$ are not difficult.

Recognition Problem:

– Is $\mathcal{I}/G$ a jet space.

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Recognition Problem:

– Is $\mathcal{I}/G$ a jet space.

– Is $\mathcal{I}/G$ the EDS for a 2nd order PDE?

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explicit calculations of $\mathcal{I}/G$ are not difficult.

Recognition Problem:

– Is $\mathcal{I}/G$ a jet space.

– Is $\mathcal{I}/G$ the EDS for a 2nd order PDE?

Theorem 1: $\mathbf{q}_G \colon (\mathcal{I}, M) \to (\mathcal{I}/G, M/G)$ is an integrable extension.

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Bäcklund Transformations by Symmetry Reduction

THEOREM 1: – Let $\mathcal{I}$ be a Pfaffian system on M with symmetry groups G_1 and G_2.
– Let H be a common subgroup of G_1 and G_2.

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Bäcklund Transformations by Symmetry Reduction

Theorem 1: – Let $\mathcal{I}$ be a Pfaffian system on M with symmetry groups G_1 and G_2.
– Let H be a common subgroup of G_1 and G_2.
– Assume M/H, M/G_1 and M/G_2 are smooth manifolds with smooth quotient maps
– Assume the actions of G_1 and G_2 are transverse to $\mathcal{I}$.

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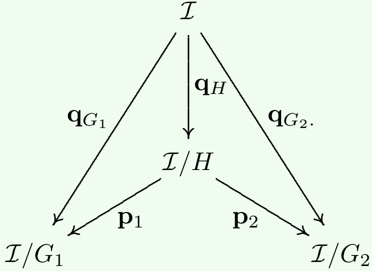
THEOREM 1: – Let $\mathcal{I}$ be a Pfaffian system on M with symmetry groups G_1 and G_2.

– Let H be a common subgroup of G_1 and G_2.

– Assume M/H, M/G_1 and M/G_2 are smooth manifolds with smooth quotient maps

– Assume the actions of G_1 and G_2 are transverse to $\mathcal{I}$.

– Then



is a commutative diagram of integrable extensions.

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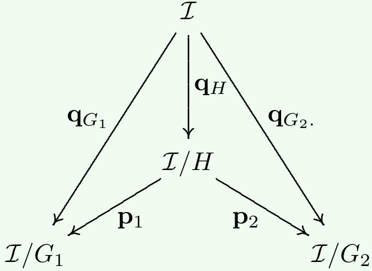

is a commutative diagram of integrable extensions.

– $\mathcal{I}/H$ is Bäcklund transformation between $\mathcal{I}/G_1$ and $\mathcal{I}/G_2$

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Example Part 3

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx,$$
$$dw - w_1\,dy,\ dw_1 - w_2\,dy,\ dw_2 - w_3\,dy\rangle$$

$$X_1 = \partial_v + \partial_w, X_2 = v\,\partial_v + w\,\partial_w, X_3 = \partial_v + \partial_w, X_4 = v^2\partial_v + w^2\partial_w$$

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$$\boxed{J^3\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R})}$$

$$\Big\downarrow \{X_1, X_2\}$$

$$\boxed{V_y = -e^W, W_x = -e^V}$$

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$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx,$$
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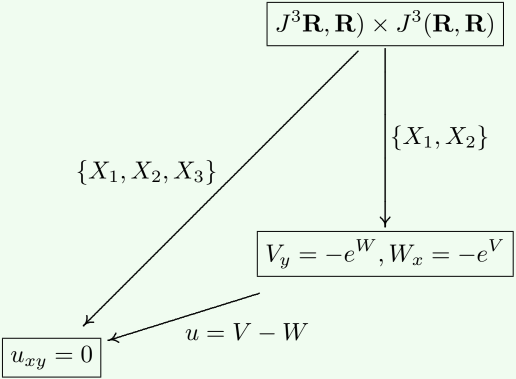

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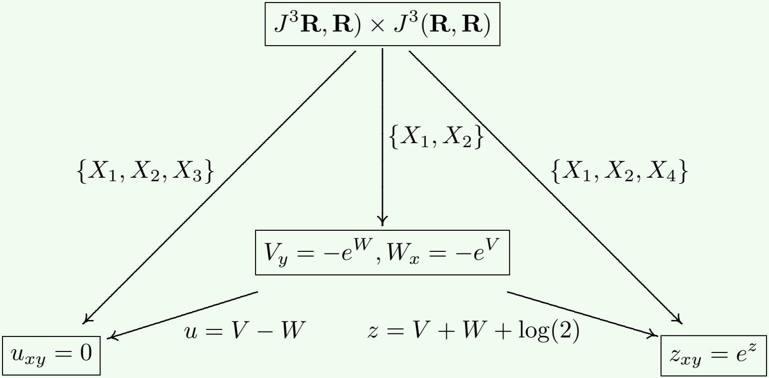
$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx,\ dw - w_1\,dy,\ dw_1 - w_2\,dy,\ dw_2 - w_3\,dy\rangle$$

$$X_1 = \partial_v - \partial_w, X_2 = v\,\partial_v + w\,\partial_w, X_3 = \partial_v + \partial_w), X_4 = v^2\partial_v + w^2\partial_w$$



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The Method of Darboux

The method of Darboux arose from the methods of:
– Laplace
– Lagrange-Charpit
– Jacobi-Mayer Method
– Ampere

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It is a theory of compatibility –

> Add more and more equations to a given system until the result is Frobenius

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The Method of Darboux

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– Ampere

It is a theory of compatibility –

> Add more and more equations to a given system until the result is Frobenius

Systems are Darboux integrable if the general solution can be found in this way.

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Precise Definition in the Classical Setting

A hyperbolic PDE in the plane

$$F(x, y, u, p, q, r, s, t) = 0$$

determines a rank 3 Pfaffian system $I = \{\theta^0, \theta^1, \theta^2\}$ on a 7 manifold.

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$$d\,\theta^0 \equiv 0, \quad d\theta^1 \equiv \pi^1 \wedge \pi^2, \quad d\theta^2 \equiv \sigma^1 \wedge \sigma^2.$$

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$$d\,\theta^0 \equiv 0, \quad d\theta^1 \equiv \pi^1 \wedge \pi^2, \quad d\theta^2 \equiv \sigma^1 \wedge \sigma^2.$$

The $\{$***singular, Monge, characteristics***$\}$ systems are

$$V_1 = \{\theta^0, \theta^1, \theta^2, \pi^1, \pi^2\} \quad V_2 = \{\theta^0, \theta^1, \theta^2, \sigma^1, \sigma^2\}.$$

A PDE in the plane is ***Darboux integrable*** if there are functions I_1, I_2, J_1, J_2 with

$$dI_1, dI_2 \in V_1 \quad dJ_1, dJ_2 \in V_2.$$

I_1, I_2 – ***Darboux Invariants, Intermediate Integrals, First Integrals of the Characteristic Systems***

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Example Part 4

LIOUVILLE EQUATION: $u_{xy} = e^u$

$$I = \{\theta^1 = du - p\,dx - q\,dy,\ \theta^2 = dp - r\,dx - e^u\,dy,\ \theta^3 = dq - e^u\,dx - q\,dy\}$$

$$V_1 = \{\theta^1,\ \theta^2,\ \theta^3,\ dx,\ dr - pe^u\,dy\}$$
$$V_2 = \{\theta^1,\ \theta^2,\ \theta^3,\ dy,\ dt - qe^u\,dx\}.$$

First integrals for V_1: $I^1 = x$, $I^2 = r - p^2/2$.

First integrals for V_2: $J^1 = y$, $J^2 = t - q^2/2$.

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Example Part 4

Liouville equation: $u_{xy} = e^u$

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$$V_1 = \{\theta^1,\ \theta^2,\ \theta^3,\ dx,\ dr - pe^u\,dy\}$$
$$V_2 = \{\theta^1,\ \theta^2,\ \theta^3,\ dy,\ dt - qe^u\,dx\}.$$

First integrals for V_1: $I^1 = x$, $I^2 = r - p^2/2$.

First integrals for V_2: $J^1 = y$, $J^2 = t - q^2/2$.

The method of Darboux:

$$u_{xx} = \frac{1}{2}u_x^2 + f(x), \quad u_{xy} = e^u, \quad u_{yy} = \frac{1}{2}u_y^2 + g(y)$$

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Liouville equation: $u_{xy} = e^u$

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$$V_1 = \{\theta^1,\ \theta^2,\ \theta^3,\ dx,\ dr - pe^u\,dy\}$$
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Question: What kind of ODE are these?

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Question: What kind of ODE are these?
Answer: (Vessiot)

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Generalizations

The classical definition of Darboux integrability can be generalized to a broad class of EDS
A Pfaffian system $I = \{\theta_0^a, \theta_1^b, \theta_2^c\}$ is Darboux integrable if
The structure equations are, $\mod I$

$$d\,\theta_0^a \equiv 0, \quad d\theta_1^b \equiv P_{ij}^b \pi^i \wedge \pi^j, \quad d\theta_2^c \equiv Q_{hk}^c \sigma^h \wedge \sigma^k$$

and if the associated systems

$$V_1 = \{\theta_0^a, \theta_1^b, \theta_2^c\, \pi^i\} \quad V_2 = \{\theta_0^a, \theta^c, \theta_2^c, \sigma^h\}$$

have *enough* first integrals.

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have *enough* first integrals.

THEOREM. An integrable extension of a Darboux integrable system is Darboux integrable.

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Darboux Integrability and Symmetry Reduction

- Let$\mathcal{K}_1$ and $\mathcal{K}_2$ be EDS on manifolds M_1 and M_2, with $(K_a^1)^\infty = \{0\}$.
- Let G a symmetry group of $\mathcal{K}_1$ and $\mathcal{K}_2$, acting transversally
- Let G act freely
- Let G_{diag} be the diagonal action on $M_1 \times M_2$, acting regularly

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THEOREM A: – $(\mathcal{K}_1 + \mathcal{K}_2)/G_{\mathrm{diag}}$ is Darboux integrable EDS

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THEOREM A: – $(\mathcal{K}_1 + \mathcal{K}_2)/G_{\mathrm{diag}}$ is Darboux integrable EDS

THEOREM B: – If $\mathcal{I}$ is Darboux integrable then

$$\mathcal{I}_U \cong (\mathcal{K}_1 + \mathcal{K}_2)/G_{\mathrm{loc,diag}}$$

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The structure constants for the Lie algebra of G are read-off from the structure equations of $\mathcal{I}$ after a series of co-frame adaptations.

These theorems give a new interpretation to the work of Vessiot.

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- Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be EDS on manifolds M_1 and M_2, with $(K_a^1)^\infty = \{0\}$.
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This local canonical quotient representation of Darboux integrable systems has been computed for many examples.

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Bäcklund Transformations for Darboux Integrable Systems

– Let $\mathcal{I}$ be a Darboux integrable with quotient representation $(\mathcal{K}_1 \times \mathcal{K}_2)/G_{\text{diag}}$.
– Let $L \subset G \times G$ be a subgroup, let $H_{\mathrm{diag}} = L \cap G_{\mathrm{diag}}$.

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Bäcklund Transformations for Darboux Integrable Systems

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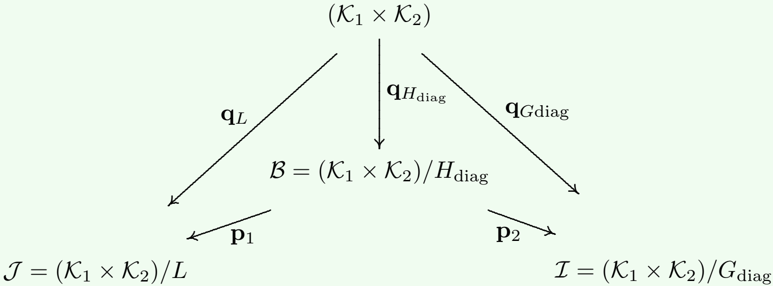

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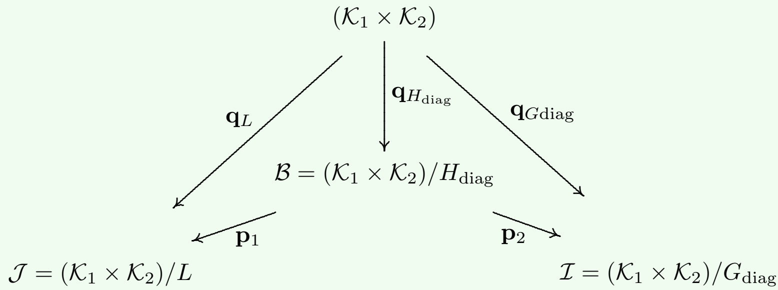

$\mathcal{J}$ always has more intermediate integrals than $\mathcal{I}$

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Example 1

Some Classical Example's from Goursat's List:

$$V_y = -e^W, W_x = -e^V$$

$\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$

$$u^1_{xy} = 0, \qquad u^2_{xy} = \frac{1}{(u^2 - x)}\, u^2_x u^2_y \qquad u^3_{xy} = \frac{2u^3 - x - y}{(u^3 - x)(u^3 - y)}\, u^3_x u^3_y \qquad u^4_{xy} = e^{u^4}$$

Lots of Bäcklund Transformations

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Example 2

We calculate Bäcklund transformations from

$$u' = F(v'') \quad y' = G(z'')$$

$$X_1 = \partial_v, \quad X_2 = s\partial_v + \partial_{v'}, \quad X_3 = \partial_u,$$

$$Y_1 = \partial_z, \quad Y_2 = t\partial_z + \partial_{z'} \quad Y_3 = \partial_y,$$

$$Z_1 = X_1 - X_2, \quad Z_2 = X_2 + Y_2, \quad Z_3 = X_3 - Y_3.$$

$$du - F(v_2)\,ds,\ dv - v_1\,ds,\ dv_1 - v_2\,ds$$
$$dy - G(z_2)\,dt,\ dz - z_1\,dt,\ dz_1 - z_2\,dt$$

$\downarrow \{Z_1, Z_2\}$

$$\beta^1 = dx_5 - x_3\,dx_1 + x_4\,dx_2$$
$$\beta^2 = dx_6 - F\,dx_2,\ \beta^3 = dx_7 - G\,dx_1$$
$$\beta^4 = dx_8 - x_5\,dx_1 + x_4(x_1 + x_2)\,dx_2$$

$\{X_1, Y_1, Z_2\}$, $\mathbf{p}_1$

$$\alpha^1 = dx_5 - x_3\,dx_1 + x_4\,dx_2,$$
$$\alpha^2 = dx_6 - F\,dx_2,\ \alpha^3 = dx_7 - G\,dx_1$$

$\{Z_1, Z_2, Z_2\}$, $\mathbf{p}_2$

$$\gamma^1 = dx_5 - x_3 dx_1 + x_4 dx_2$$
$$\gamma^2 = dx_6 - G\,dx_1 - F\,dx_2$$
$$\gamma^3 = dx_8 - x_5 dx_1 + x_4(x_1 + x_2) dx_2$$

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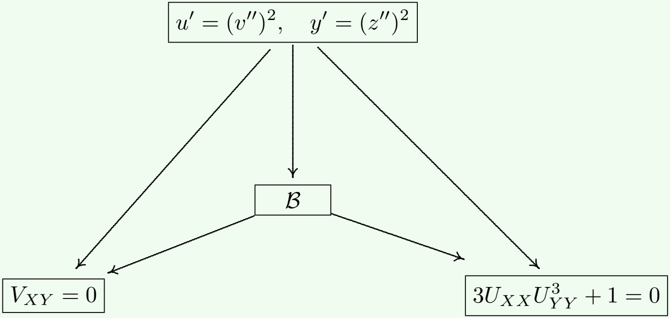
For the Hilbert-Cartan equations

$$u' = (v'')^2 \quad y' = (z'')^2$$

this yields



NON MONGE-AMPERE EXAMPLE

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Example 3

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The Goursat Equation:

A Bäcklund transformation between

$$V_{xy} = \frac{2(n-1)\sqrt{V_x V_y}}{x+y} \quad \text{and} \quad U_{xy} = \frac{2n\sqrt{U_x U_y}}{x+y}$$

is

$$(\sqrt{U_x} - \sqrt{V_x})^2 + \frac{(2n+1)(U-V)}{x+y} = 0$$
$$\sqrt{U_y} + \sqrt{U_y} = -\sqrt{V_x} + \sqrt{U_x}.$$

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Example 3

THE GOURSAT EQUATION:

A Bäcklund transformation between

$$V_{xy} = \frac{2(n-1)\sqrt{V_x V_y}}{x+y} \quad \text{and} \quad U_{xy} = \frac{2n\sqrt{U_x U_y}}{x+y}$$

is

$$(\sqrt{U_x} - \sqrt{V_x})^2 + \frac{(2n+1)(U-V)}{x+y} = 0$$

$$\sqrt{U_y} + \sqrt{U_y} = -\sqrt{V_x} + \sqrt{U_x}.$$

(An example of an equation integrable at higher orders)

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The A_2 Toda Lattice

$$J^{3,4} \times J^{3,4}$$

$\{\Gamma, Z_3, Z_4\}$ $\qquad$ Γ $\qquad$ $\{\Gamma, Z_1, Z_2\}$

$$W^1_{xy} = -W^1_y(W^2_x + e^{W_1})$$
$$W^2_{xy} = -W^2_x(e^{W^2} + W^1_y)$$

$\mathbf{p}_1$ $\qquad$ $\mathbf{p}_2$

$$V^1_{xy} = e^{V^1}$$
$$V^2_{xy} = 0$$

$$U^1_{xy} = e^{2U^1 - U^2}$$
$$U^2_{xy} = e^{-U^1 + 2U^2}$$

Example 4

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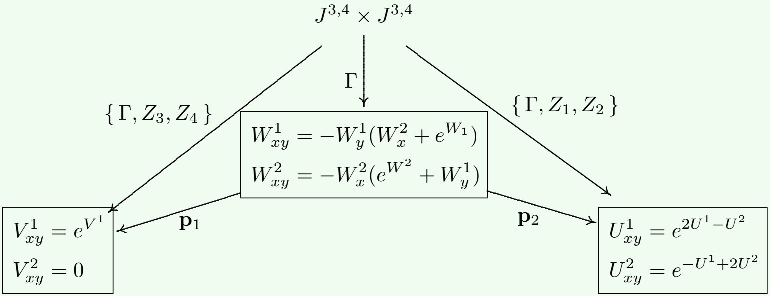
The A_2 Toda Lattice



An Interesting Conjecture?

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Example 5

The equation

$$u_{xy} = \frac{\sqrt{1 - u_x^2}\,\sqrt{1 - u_y^2}}{\sin u}$$

has $so(3)$ as its Vessiot algebra.

It cannot be ***real*** Bäcklund equivalent (with one-dimensional fiber) to the wave equation.

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Idea:

– The differential system $\mathcal{B}$ defining the Bäcklund transformation is DI with 2 dim. Vessiot alg.

– By a uniqueness theorem, $\mathcal{B}$ must come from the group-theoretical construction discussed here.

– The Vessiot algebra of $\mathcal{B}$ is a 2-dimensional sub-algebra of $so(3)$.

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IDEA:

– The differential system $\mathcal{B}$ defining the Bäcklund transformation is DI with 2 dim. Vessiot alg.

– By a uniqueness theorem, $\mathcal{B}$ must come from the group-theoretical construction discussed here.

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This is at odds with a result of Clelland-Ivey.

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Example 6

The sine-Gordon equation is not Darboux integrable at any order of prolongation (a direct proof was given by Lie).

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Example 6

The sine-Gordon equation is not Darboux integrable at any order of prolongation (a direct proof was given by Lie).

We are working on a new (shorter?) proof using the fact that the sine-Gordon equation has an auto-Bäcklund transformation

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Summary

In the geometric theory of differential equations, one likes to study

– symmetries (and generalized symmetries)

– conservation laws (and characteristic cohomology)

– transformation theory (for example, Bäcklund transformations)

– equivalence problems

– solution methods

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In all these subjects Darboux integrable equations naturally arise as very special equations.

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Thank-you !

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