THE AFFINE BONNET PROBLEM

ROBERT L. BRYANT

MATHEMATICAL SCIENCES RESEARCH INSTITUTE

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1. Bonnet and Euclidean Surface Theory

Let M^2 be oriented and $(u, v) : U \to \mathbb{R}^2$ oriented local coords.
Let $\mathbf{x} : M^2 \to \mathbb{E}^3$ be an immersion. The *first fundamental form* is

$$I_{\mathbf{x}} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}.$$

It does not determine $\mathbf{x}$ up to Euclidean motion. A first-order equivariant map $u_{\mathbf{x}} : M^2 \to S^2$ is the *Gauss map*

$$u_{\mathbf{x}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|},$$

which allows one to define the *second fundamental form*

$$II_{\mathbf{x}} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}u_{\mathbf{x}}.$$

Bonnet Uniqueness: The pair $(I_{\mathbf{x}}, II_{\mathbf{x}})$ determine $\mathbf{x} : M^2 \to \mathbb{E}^3$ up to Euclidean motion.

Too much information? $\mathbf{x}$ is locally 3 functions of 2 variables, but $(I_{\mathbf{x}}, II_{\mathbf{x}})$ is locally 6 functions of 2 variables.

Compatibility: For balance, there should be 3 relations, and there are:

$$\det_{I_{\mathbf{x}}}\big(I\!I_{\mathbf{x}}\big) = K(I_{\mathbf{x}}) \qquad \text{(Gauss, 1 equation)}$$

$$\delta_{I_{\mathbf{x}}}\big(I\!I_{\mathbf{x}}\big) = 0 \qquad \text{(Codazzi, 2 equations).}$$

where $\delta_I : C^\infty\big(S^2(T^*)\big) \to C^\infty\big(T^*\big)$ is a first order linear operator, defined for any $I > 0$.

Bonnet Existence: If M^2 is 1-connected and $I > 0$ and $I\!I$ are quadratic forms on M that satisfy

$$\det_I\big(I\!I\big) = K(I) \qquad \text{and} \qquad \delta_I\big(I\!I\big) = 0,$$

then there exists an immersion $\mathbf{x} : M \to \mathbb{E}^3$ such that $(I, I\!I) = (I_{\mathbf{x}}, I\!I_{\mathbf{x}})$.

Bonnet's Questions:

1. Do we need all of $(I_{\mathbf{x}}, I\!I_{\mathbf{x}})$ to specify $\mathbf{x}$? For example, would it be enough to know $I_{\mathbf{x}}$ and the *mean curvature* $H_{\mathbf{x}} = \frac{1}{2}\,\mathrm{tr}_{I_{\mathbf{x}}}\big(I\!I_{\mathbf{x}}\big)$?
2. What are the compatibility conditions for (I, H) in order that there exist $\mathbf{x}$ such that $(I, H) = (I_{\mathbf{x}}, H_{\mathbf{x}})$?

Assume 'no umbilics': $H^2 - K(I) = r^2 > 0$. Fix a local I-orthonormal coframing (ω_1, ω_2) and seek a function θ such that

$$I = {\omega_1}^2 + {\omega_2}^2$$

$$I\!I = (H+r\cos\theta)\,{\omega_1}^2 + 2r\sin\theta\,\omega_1\omega_2 + (H-r\cos\theta)\,{\omega_2}^2.$$

This solves the Gauss equation. The Codazzi equations take the form

$$0 = \delta_I(I\!I) = r\,(\mathrm{d}\theta - \alpha_0 - \cos\theta\,\alpha_1 - \sin\theta\,\alpha_2)$$

where the 1-forms α_i are computed from (ω_1, ω_2, H) and their derivatives.

Thus, Bonnet investigated the overdetermined system for θ

$$\mathrm{d}\theta = \alpha_0 + \cos\theta\,\alpha_1 + \sin\theta\,\alpha_2\,.$$

He asked 'When is this formally integrable?'

Taking the exterior derivative of both sides and using the equation itself, one obtains

$$0 = (A_0 + A_1\,\cos\theta + A_2\,\sin\theta)\,\omega_1\wedge\omega_2$$

where the functions A_i are computed from (ω_1, ω_2, H) and their derivatives. Formal integrability is then equivalent to $A_0 \equiv A_1 \equiv A_2 \equiv 0$.

Bonnet's Theorem: If (I, H) satisfies $A_0 \equiv A_1 \equiv A_2 \equiv 0$, then either

- H is constant and I is the first fundamental form of a surface in $\mathbb{E}^3$ with constant mean curvature H (*Associated Surfaces*), or
- Up to diffeomorphism, (I, H) belongs to a 3-parameter family of data with $\mathrm{d}H \neq 0$. (*The surfaces of Bonnet*)

There are many works on the surfaces of Bonnet (including important work by É. Cartan). They were eventually shown to be expressible in terms of ϑ-functions and Painlevé equations by Bobenko and Eitner.

Bonnet mates: If ${A_1}^2 + {A_2}^2 > {A_0}^2$, there are two solutions θ_1 and θ_2 to $0 = A_0 + A_1 \cos\theta + A_2 \sin\theta$, and, hence, at most two realizations $\mathbf{x}_1$ and $\mathbf{x}_2$ of the data (I, H). These are called *Bonnet mates*, when they exist.

É. Cartan's Theorem: The family of surfaces $\mathbf{x}(M) \subset \mathbb{E}^3$ that possess Bonnet mates depends on 4 functions of 1 variable.

Can be interpreted in terms of pseudo-holomorphic curves and Lax pairs....

2. Unimodular Affine Surface Theory

$\mathbb{A}^3$ is (unimodular) affine 3-space, acted on by translations and volume-preserving linear transformations.

Given an immersion $\mathbf{x} : M^2 \to \mathbb{A}^3$, define a second-order invariant (using local coordinates) by

$$B_{\mathbf{x}} = \det \begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v & \mathbf{x}_{uu}\,\mathrm{d}u^2 + 2\mathbf{x}_{uv}\,\mathrm{d}u\mathrm{d}v + \mathbf{x}_{vv}\,\mathrm{d}v^2 \end{pmatrix} \otimes (\mathrm{d}u \wedge \mathrm{d}v).$$

$B_{\mathbf{x}}$ is a section of the bundle $S^2(T^*) \otimes \Lambda^2(T^*)$.

Say that $\mathbf{x}$ is *locally strictly convex* if $B_{\mathbf{x}}$ is definite. In this case (assumed henceforth), M has a unique metric $I_{\mathbf{x}}$ and orientation $*_{\mathbf{x}}$ so that

$$B_{\mathbf{x}} = I_{\mathbf{x}} \otimes *_{\mathbf{x}} 1 = \left({\omega_1}^2 + {\omega_2}^2\right) \otimes \omega_1 \wedge \omega_2\,.$$

$I_{\mathbf{x}}$ is the *Blaschke metric* associated to $\mathbf{x}$; it is second-order in $\mathbf{x}$, and does NOT determine $\mathbf{x}$ up to unimodular affine equivalence (u.a.e.).

The affine normal: Writing $\mathrm{d}\mathbf{x} = \mathbf{e}_1\,\omega_1 + \mathbf{e}_2\,\omega_2$ and $\det(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) \equiv 1$ only determines $\mathbf{e}_3$ up to addition of multiples of $\mathbf{e}_1$ and $\mathbf{e}_2$.

Proposition: There is a unique $n_{\mathbf{x}} : M \to \mathbb{A}^3$ such that $\det(\mathbf{e}_1\ \mathbf{e}_2\ n_{\mathbf{x}}) \equiv 1$ and so that $\mathrm{d}n_{\mathbf{x}} \equiv 0 \mod \mathbf{e}_1, \mathbf{e}_2$. This $n_{\mathbf{x}}$ is third-order in $\mathbf{x}$.

The affine form $I\!I_{\mathbf{x}}$: Writing $\mathrm{d}n_{\mathbf{x}} = \mathbf{e}_1\,\pi_1 + \mathbf{e}_2\,\pi_2$ and setting

$$I\!I_{\mathbf{x}} = -\pi_1 \circ \omega_1 - \pi_2 \circ \omega_2$$

gives a well-defined quadratic form that is fourth-order in $\mathbf{x}$.

Surprise(?): The affine data $(I_{\mathbf{x}}, I\!I_{\mathbf{x}})$ does *not* determine $\mathbf{x}$ up to u.a.e.

Missing third order information: One finds that

$$\nabla^{I_{\mathbf{x}}}(\mathrm{d}\mathbf{x}) = \mathbf{e}_1 \otimes Q_1 + \mathbf{e}_2 \otimes Q_2 + n_{\mathbf{x}} \otimes I_{\mathbf{x}}$$

and that $C_{\mathbf{x}} = Q_1 \circ \omega_1 + Q_2 \circ \omega_2$ is a well-defined cubic form on M and is a third-order invariant of $\mathbf{x}$.

Radon Uniqueness: The pair $(I_{\mathbf{x}}, C_{\mathbf{x}})$ determine $\mathbf{x}$ up to u.a.e.

Too much information? A section of $S^2(T^*) \oplus S^3(T^*)$ defines $3+4 = 7$ functions of two variables, but $\mathbf{x}$ consists of 3 functions of two variables.

Compatibility Two sets of conditions are found to hold on the data:

$$\mathrm{tr}_{I_{\mathbf{x}}}(C_{\mathbf{x}}) = 0 \qquad \text{(apolarity)}$$

i.e., $C_{\mathbf{x}}$ is a section of a rank 2 subbundle $S^3_0(T^*, I_{\mathbf{x}}) \subset S^3(T^*)$ defined by I_x. In addition,

$$D_{I_{\mathbf{x}}}(C_{\mathbf{x}}) = 0 \qquad \text{(Radon)}$$

where $D_I : C^\infty\left(S^3_0(T^*, I)\right) \to C^\infty(T^*)$ is a nonlinear, second-order elliptic operator (that depends on a metric I and choice of orientation).

Radon Existence: If M^2 is 1-connected, oriented, and $I > 0$ and C are a quadratic and cubic form on M that satisfy

$$\mathrm{tr}_I\big(C\big) = 0 \qquad \text{and} \qquad D_I(C) = 0,$$

then there exists an immersion $\mathbf{x} : M \to \mathbb{A}^3$ such that $(I, C) = (I_{\mathbf{x}}, C_{\mathbf{x}})$.

Will less information do? Since $H_{\mathbf{x}} = \mathrm{tr}_{I_{\mathbf{x}}}\left(I\!I_{\mathbf{x}}\right) = K(I_{\mathbf{x}}) - 2\left|C_{\mathbf{x}}\right|^2_{I_{\mathbf{x}}}$, the data $(I_{\mathbf{x}}, H_{\mathbf{x}})$ is equivalent to the data $I_{\mathbf{x}}$ and $2\left|C_{\mathbf{x}}\right|^2_{I_{\mathbf{x}}}$ (aka the *Pick invariant*).

The affine Bonnet Problem: Determine when, and in how many ways (I, H) can be realized as $(I_{\mathbf{x}}, H_{\mathbf{x}})$ for some $\mathbf{x} : M \to \mathbb{A}^3$.

Assume 'no umbiilics': Write $K(I) - H = 2r^2 > 0$. Then, fixing an I-orthonormal, oriented coframe (ω_1, ω_2), seek a function θ such that

$$I = {\omega_1}^2 + {\omega_2}^2$$

$$C = r\cos\theta\ \left({\omega_1}^3 - 3\,\omega_1{\omega_2}^2\right) - r\sin\theta\ \left({\omega_2}^3 - 3\,\omega_2{\omega_1}^2\right).$$

will satisfy Radon's condition $D_I(C) = 0$, which is then two second-order equations for θ.

In local coordinates with $I = F\,|\mathrm{d}z|^2$, Radon's condition takes the form

$$\theta_{zz} = E(\theta, \theta_z)$$

for an expression E that depends on I and H. If it were formally integrable, one could specify $(\theta, \theta_z, \theta_{z\bar{z}})$ at a point of M and have a unique (local) solution.

Theorem 1: (B—,2009) The equation $D_I(C) = 0$ is never formally integrable for any pair (I, H).

Theorem 2: (B—,2009) There exist data (I, H) such that the equation $D_I(C) = 0$ has a 3-parameter family of solutions. These data depend on 2 arbitrary functions of 1 variable, and the integration of the corresponding system $D_I(C) = 0$ can be linearized as an integrable system.

Application: The surfaces with K and H constant, studied by Vrancken and Dillen, et al, can now be explicitly constructed.

Theorem 3: (B—,2009) The surfaces $\mathbf{x}(M) \subset \mathbb{A}^3$ that have at least one affine Bonnet mate depend on 7 arbitrary functions of 1 variable.

Remark: There do exist surfaces $\mathbf{x}(M) \subset \mathbb{A}^3$ that have exactly a 1-parameter family of affine Bonnet mates. I do not know how many such families there are or whether an exactly 2-parameter family of affine Bonnet mates is possible.