Value distribution of the Gauss map of improper affine fronts and affine Bernstein problem

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[1] (Intro.) Value distribution of Gauss map

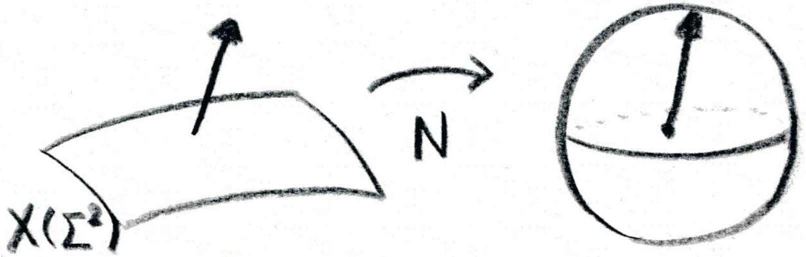
$X : \Sigma^2 \to \mathbf{R}^3$ minimal surface ($\Leftrightarrow$ mean curv. $H \equiv 0$)

Theorem. (S. N. Bernstein, 1915)

$\Sigma^2 = \mathbf{R}^2$, $X(u,v) = (u, v, f(u,v))$ minimal graph
Then $X(\Sigma^2)$ is a plane (i.e. $f(u,v) = au + bv + c$).

$N : \Sigma^2 \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$ its Gauss map

- N is bounded on $\mathbf{R}^2 = \mathbf{C}$.
- N is constant $\Leftrightarrow$ X is a plane.



The Liouville and Bernstein thms are closely related.

※ (u,v) : isothermal coord. $\Rightarrow$ N is a mero. fct. on a Riem. surf. Σ^2.

Theorem. (Fujimoto, 1988)

$X : \Sigma^2 \to \mathbf{R}^3$ complete non-flat minimal surface

$N : \Sigma^2 \to \mathbf{C} \cup \{\infty\}$ its Gauss map

$D_N := \sharp(\mathbf{C} \cup \{\infty\} \backslash N(\Sigma^2))$: the number of exceptional values of N

Then

$$D_N \leq 4.$$

※ If the metric $ds^2 = \langle dX, dX \rangle$ is complete, X is called complete.

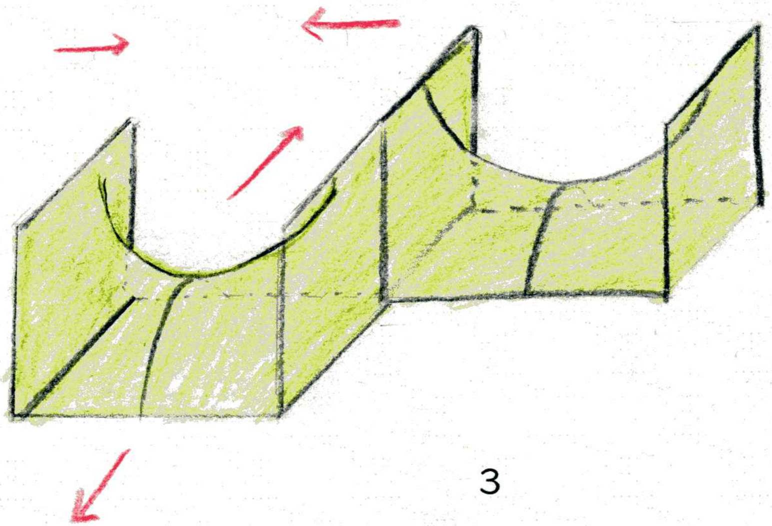
※ The inequality is sharp.

<u>**Ex.**</u> The Voss surface (The Scherk surface)

$\Sigma^2 =$ the universal cover of $\mathbf{C} \backslash \{a_1, a_2, a_3\}$

W-data $(\omega, N) = \left(\dfrac{dz}{\prod_j (z - a_j)}, z \right)$

Then $D_N = 4$.



Bernstein type results (parametric form)

- Any affine complete improper affine sphere must be an elliptic paraboloid. (Jörgens, Calabi)
- Any cplt flat surf. in $\mathbf{H}^3$ must be a horosphere or hyperbolic cylinder. (Sasaki, Volkov-Vladimirova)

Point

If we consider the classes with some admissible singularities (for example, front), then

Bernstein type result $\Leftrightarrow$ Lioville property for Gauss map

[2] Preliminaries

Definition (Martínez, 2005)

A smooth map $\psi = (\boldsymbol{x}, \varphi)\colon \Sigma^2 \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ is an improper affine front (improper affine map) if there exists a special Lagrangian imm. $L_\psi := \boldsymbol{x} + \sqrt{-1}\boldsymbol{n}\colon \Sigma^2 \to \mathbf{C}^2$ s.t.

$$\psi = \left(\boldsymbol{x}, -\int \langle \boldsymbol{n}, d\boldsymbol{x} \rangle\right).$$

※ An IA-front is a front in $\mathbf{R}^3$. (Nakajo, Umehara-Yamada)

※ IA-spheres = IA-fronts with no singularity

(= Blaschke immersion with its shape operator $S \equiv 0$)

IA-fronts = IA-spheres with some admissible singularities

Fact.

- Every SL-imm. is a minimal L-imm. (see Harvey-Lowson)
- Every mini. L-imm. in $\mathbf{C}^2$ is a complex curve. (Chen-Morvan)

For an IA-front $\psi = (\boldsymbol{x}, \varphi)\colon \Sigma^2 \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$,
there exists a complex curve $\alpha\colon \Sigma^2 \to \mathbf{C}^2$, $\alpha := (F, G)$ s.t.

$$\boldsymbol{x} = G + \bar{F}, \quad \boldsymbol{n} = \bar{F} - G\,.$$

Then, the flat fundamental form

$$ds^2 = \langle d\boldsymbol{x}, d\boldsymbol{x}\rangle = |dF + dG|^2$$

and the induced metric of L_ψ from $\mathbf{C}^2$

$$\begin{aligned} d\tau^2 &= \langle d\boldsymbol{x}, d\boldsymbol{x}\rangle + \langle d\boldsymbol{n}, d\boldsymbol{n}\rangle \\ &= 2(|dF|^2 + |dG|^2)\,. \end{aligned}$$

Complex representation (Martínez, 2005)

Σ^2 : a Riemann surface

(F, G) : a pair of holomorphic functions on Σ^2 s.t.

(1) $\forall \gamma \in H_1(\Sigma^2, \mathbf{Z})$, $\mathrm{Re} \int FdG = 0$,

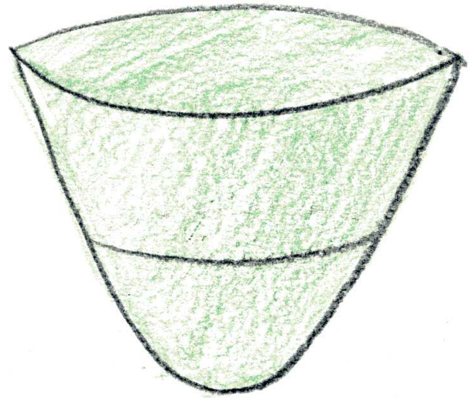
(2) $2(|dF|^2 + |dG|^2)$ is positive definite.

Then the map $\psi\colon \Sigma^2 \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ given by

$$\psi := \left(G + \bar{F}, \frac{|G|^2 - |F|^2}{2} + \Re\left(GF - \int FdG \right) \right)$$

is an IA-front in $\mathbf{R}^3$. Conversely, any IA-front is given in this way. The singular pts of ψ correspond with the pts where $|dF| = |dG|$.

● Examples of IA-fronts



Ex. elliptic paraboloids

$\Sigma^2 = \mathbf{C}$

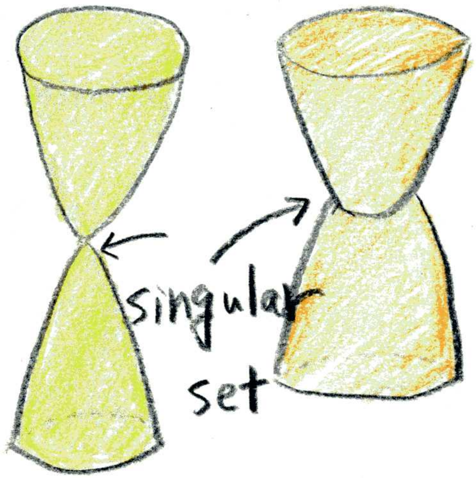
W-data $(F, G) = (z, kz)$ (k: constant)

This is also an IA-sphere (i.e. with no singularity).

Ex. rotational IA-fronts (Martínez, 2005)

$\Sigma^2 = \mathbf{C}\backslash\{0\}$

W-data $(F, G) = \left(z, \pm\dfrac{r^2}{z}\right)$ ($r \in \mathbf{R}\backslash\{0\}$)



Definition (Martínez, 2005)

The meromorphic function $\nu\colon \Sigma^2 \to \mathbf{C} \cup \{\infty\}$ given by

$$\nu := \frac{dF}{dG}$$

is called Lagrangian Gauss map of ψ.

※ The singular pts of ψ correspond with the pts where $|\nu| = 1$.

Geometric meaning of ν

$L_\psi\colon \Sigma^2 \to \mathbf{R}^4 (\simeq \mathbf{C}^2)$: a special Lagrangian lift of ψ

$\mathcal{G}\colon \Sigma^2 \to (\mathbf{C} \cup \{\infty\}) \times (\mathbf{C} \cup \{\infty\})$: the Gauss map of $L_\psi(\Sigma^2)$ in $\mathbf{R}^4$

$$\Rightarrow \qquad \mathcal{G} = \left(1, \frac{dF}{dG}(= \nu)\right) \in (\mathbf{C} \cup \{\infty\}) \times (\mathbf{C} \cup \{\infty\})$$

[3] Main results: Value distribution of ν for weakly cplt

Definition (Umehara-Yamada, 2011)

An IA-front is called weakly complete if the induced metric $d\tau^2 = 2(|dF|^2 + |dG|^2)$ is a complete Riemannian metric on Σ^2.

Main theorem I (K-Nakajo, 2011)

$\psi\colon \Sigma^2 \to \mathbf{R}^3$: weakly complete IA-front

If ν is constant, then ψ is an elliptic paraboloid.

Main theorem II. (K-Nakajo, 2010)

$\psi\colon \Sigma^2 \to \mathbf{R}^3$ weakly complete IA-front

$\nu\colon \Sigma^2 \to \mathbf{C} \cup \{\infty\}$ its L-Gauss map

$D_\nu := \sharp(\mathbf{C} \cup \{\infty\} \backslash \nu(\Sigma^2))$: the number of exceptional values of ν

Then

$$D_\nu \leq 3.$$

※ The inequality is sharp.

Ex. Voss type of IA-front (K-Nakajo, 2010)

$\Sigma^2 =$ the universal cover of $\mathbf{C} \backslash \{a_1, a_2\}$

W-data $(dG, \nu) = \left(\dfrac{dz}{\prod_j (z - a_j)}, z \right)$

Then it is weakly complete and $D_\nu = 3$.

● Sketch of the proof of Main theorem II

- $\widetilde{\Sigma^2} = \mathbf{C} \to$ By the little Picard theorem, $D_\nu \leq 2$.
- $\widetilde{\Sigma^2} = \mathbf{D}$ (the unit disk)

$$d\tau^2 = 2(|dF|^2 + |dG|^2) = 2(1 + |\nu|^2)|dG|^2 .$$

If $D_\nu \geq 4$, then $1/2 < \exists\lambda < 1$, $\exists\Psi$: $\triangle_R = \{|z| < R\} \to \mathbf{D}$ isometry s.t.

$$\Psi^* d\tau = C^\lambda \left(\frac{R}{R^2 - |z|^2} \right)^\lambda |dz| \quad (C\text{: constant}) .$$

We set $d(p)$ = the distance from $p \in \mathbf{D}$ to $\partial\mathbf{D}$, then

$$d(p) \leq \int d\tau = \int \Psi^* d\tau = C^\lambda \int \left(\frac{R}{R^2 - |z|^2} \right)^\lambda |dz| < +\infty .$$

It contradicts that $d\tau^2$ is complete.

Main theorem III. (K-Nakajo, 2011)

$\psi\colon \Sigma^2 \to \mathbf{R}^3$ weakly complete IA-front with $\int |K_{d\tau^2}| dA_{d\tau^2} < +\infty$

$\nu\colon \Sigma^2 \to \mathbf{C} \cup \{\infty\}$ its L-Gauss map

$D_\nu := \sharp(\mathbf{C} \cup \{\infty\} \backslash \nu(\Sigma^2))$: the number of exceptional values of ν

Then

$$D_\nu \leq 2.$$

※ The inequality is also sharp.

Ex. rotational IA-front (Martínez, 2005)

$\Sigma^2 = \mathbf{C} \backslash \{0, \infty\}$

W-data $(dG, \nu) = \left(\dfrac{r^2}{z^2} dz, \dfrac{z^2}{r^2} \right)$ $(r \in \mathbf{R} \backslash \{0\})$

Then it is weakly complete and $D_\nu = 2$.

● Application : the Bernstein type theorem for IA-spheres

Corollary (Jörgens 1954, Calabi 1958)

Any affine cplt IA-sphere must be an elliptic paraboloid.

※ (The affine metric of IA-front) $h := |dG|^2 - |dF|^2$

Proof.

Because an IA-sphere has no singularities, it holds that $|\nu| < 1$.

On the other hand, we have

$$h = |dG|^2 - |dF|^2 < 2(|dF|^2 + |dG|^2) = d\tau^2.$$

Thus, if an IA-sphere is affine cplt, then it is also weakly complete.

By the Main results I and II, it is an elliptic paraboloid.

[4] Further topic : Affine Bernstein problem

$\psi\colon \Sigma^2 \to \mathbf{R}^3$ an affine immersion, S: its affine shape operater,

- affine maximal surfaces (AM-surfaces) $\cdots$ $H_A := \mathrm{tr}(S)/2 \equiv 0$.
- improper affine spheres (IM-spheres) $\cdots$ $S \equiv 0$.

$\Rightarrow$ { improper affine sphere } $\subset$ { affine maximal surface }

The parametric affine Bernstein problem (proposed by Calabi)

A locally strongly convex, affine cplt, AM-surfs in $\mathbf{R}^3$ is an elliptic paraboloid.

※ This is solved by Trudinger-Wang and A. M. Li and F. Jia (2002).
※ The previous result is the special case of this problem.

References

- Y. Kawakami and D. Nakajo, *"Value distribution of the Gauss map of improper affine spheres"*, preprint, arXiv:1004.1484.

- A. Martínez, *"Improper affine maps"* , Math. Z. **249**, 755–766, (2005).