The lightlike geometry of spacelike submanifolds in Minkowski space

joint work with

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§1 Gaussian Differential Geometry; brief review

- $X : U \longrightarrow \mathbb{R}^n$: an embedding (hypersurface) ($U \subset \mathbb{R}^{n-1}$: open, $X(U) = M$.)
- $n(u)$:*Unit normal vector* at $p = X(u)$.
- The *Gauss map* : $G_M : U \longrightarrow S^{n-1}$: $G_M(u) = n(u)$
- $S_p = -dG_M(u) : T_pM \longrightarrow T_pM$: the *shape operator*.
- The *Gauss-Kronecker curvature* : $K(p) = \det S_p$.
- The *mean curvature* : $H(p) = \dfrac{1}{n-1}\mathrm{Trace}\, S_p$.
- Related results on hypersurfaces: The Gauss-Bonnet theorem, The Weierstrass representation formula for a minimal surface, etc.
- For a general submanifolds $M^s \subset \mathbb{R}^n$, consider the unit normal bundle $N_1(M)$.
- The *(generalized) Gauss map* : $G_{N_1(M)} : N_1(M) \longrightarrow S^{n-1}$: $G_{N_1(M)}(p,\xi) = \xi$
- The *Lipschitz-Killing curvature* at (p,ξ) : $K(p,\xi)$ (can be defined).
- The *total (absolute) curvature* of M at p : $K^*(p) = \displaystyle\int_{\xi\in N_1(M)_p} |K(p,\xi)| d\sigma_{n-s-1}$.
- The related reults: The Chern-Lashof theorem, Convexity, Tightness etc.

§2 Lorentz-Minkowski sapce:$\mathbb{R}^{n+1}_1$

- $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle , \rangle)$: Lorentz-Minkowski $n+1$-space
- $\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$, where $x = (x_0, x_1, \dots, x_n), y = (y_0, y_1 \dots, y_n)$
- $x \in \mathbb{R}^{n+1}_1 \setminus \{0\}$ is

$$\begin{cases} \textit{spacelike} & \text{if} \quad \langle x, x\rangle > 0 \\ \textit{lightlike} & \text{if} \quad \langle x, x\rangle = 0 \\ \textit{timelike} & \text{if} \quad \langle x, x\rangle < 0, \end{cases}$$

- For any lightlike vector x, define

$$\widetilde{x} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{x = (x_0, x_1, \dots, x_n) \mid \langle x, x\rangle = 0,\ x_0 = 1\}.$$

- *Hyperplane with pseudo normal* n : $HP(n, c) = \{x \in \mathbb{R}^{n+1}_1 | \langle x, n\rangle = c\}$ for any $n \in \mathbb{R}^{n+1}_1 \setminus \{0\}$ and $c \in \mathbb{R}$.
- $HP(n, c)$ is

$$\begin{cases} \textit{spacelike} & \text{if} \quad n \text{ is timelike} \\ \textit{lightlike} & \text{if} \quad n \text{ is ligtlike} \\ \textit{timelike} & \text{if} \quad n \text{ is spacelike}, \end{cases}$$

- ***Pseudo-spheres*** in $\mathbb{R}^{n+1}_1$:

$$\begin{cases} H^n(-1) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = -1\} : \textit{Hyperbolic n-space} \\ S^n_1 = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = 1\} : \textit{de Sitter n-space} \\ LC^* = \{x \neq 0 \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = 0\} : \textit{(open) lightcone} \end{cases}$$

- For any $x_1, x_2, \dots, x_n \in \mathbb{R}^{n+1}_1$,

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \begin{vmatrix} -e_0 & e_1 & \cdots & e_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix}, \quad x_i = (x^i_0, x^i_1, \dots, x^i_n).$$

- $x_1 \wedge x_2 \wedge \cdots \wedge x_n$: pseudo orthogonal to any x_i $(i = 1, \dots, n)$.
- We choose $e_0 = (1, 0, \dots, 0)$ as the future timelike vector field.

§3 Spacelike submanifolds with codimension two

- $X : U \longrightarrow \mathbb{R}^{n+1}_1$: a spacelike embedding ($U \subset \mathbb{R}^{n-1}$: open, $M = X(U)$.)
- T_pM : a spacelike subspace at any point $p = X(u) \in M$.
- N_pM : the pseudo-normal space, a timelike plane (i.e., Lorentz plane).
- $N(M)$: the pseudo-normal bundle over M.
- $n^T(u) \in N_p(M)$: arbitrarily *future directed unit timelike normal vector field*.
- $n^S(u) \in N_p(M)$: the *spacelike unit normal vector field* defined by

$$n^S(u) = \frac{n^T(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)}{\|n^T(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)\|}$$

- $\{n^T, n^S\}$: a *pseudo-orthonormal frame field* along M.
- $d(n^T \pm n^S)_u : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M)$: linear mapping.
- Consider the psudo-orthogonal projection : $\pi^t : T_pM \oplus N_p(M) \longrightarrow T_pM$
- $S_p(n^T, \pm n^S) = -\pi^t \circ d(n^T \pm n^S)^t_u : T_pM \longrightarrow T_pM$: *$(n^T, \pm n^S)$-shape operator*

§3 Spacelike submanifolds with codimension two

- The ***lightcone Gauss-Kronecker curvature*** with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$:

$$K_\ell^\pm(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \det S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S).$$

- The ***lightcone mean curvature*** with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$:

$$H_\ell^\pm(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \frac{1}{n-1}\mathrm{Trace}\, S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S).$$

- $\overline{\boldsymbol{n}}^T(u)$: another future pointed nomal vector field; $(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S}) = (\widetilde{\overline{\boldsymbol{n}}^T \pm \overline{\boldsymbol{n}}^S}) \in S_+^{n-1}$.
- The ***lightcone Gauss map*** : $\widetilde{\mathbb{L}}^\pm : U \longrightarrow S_+^{n-1}$: $\widetilde{\mathbb{L}}^\pm(u) = (\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(u)$
- $\widetilde{S}_p^\pm = -\pi^t \circ d\widetilde{\mathbb{L}}_p^\pm : T_pM \longrightarrow T_pM$: the ***normalized lightcone shape operator***.
- The ***normalized lightcone Gauss-kronecker curvature*** : $\widetilde{K}_\ell^\pm(u) = \det \widetilde{S}_p^\pm$.
- The ***normalized lightcone mean curvature*** : $\widetilde{H}_\ell^\pm(u) = \dfrac{1}{n-1}\mathrm{Trace}\, \widetilde{S}_p^\pm$.

Proposition (The relation between curvatures)

$$\widetilde{K}_\ell^\pm(u) = \left(\frac{1}{\ell_0^\pm(u)}\right)^{n-1} K_\ell^\pm(n^T, n^S)(u), \quad \widetilde{H}_\ell^\pm(u) = \left(\frac{1}{\ell_0^\pm(u)}\right) H_\ell^\pm(n^T, n^S)(u),$$

where $(n^T \pm n^S)(u) = (\ell_0^\pm(u), \ell_1^\pm(u), \dots, \ell_n^\pm(u))$.

Corollary

(1) $\widetilde{K}_\ell^\pm(u) = 0$ *if and only if* $K_\ell^\pm(n^T, n^S)(u) = 0$.

(2) $\widetilde{H}_\ell^\pm(u) = 0$ *if and only if* $H_\ell^\pm(n^T, n^S)(u) = 0$.

- The flatness is independent of the choice of $n^T(u)$.

Proposition

Suppose $n = 3$. *Let* $\mathfrak{H}$ *be the mean curvature vector along* M. *Then*

$$\widetilde{H}_\ell^\pm \equiv 0 \Leftrightarrow \mathfrak{H} : \ lightlike \Leftrightarrow M : \ a \ marginary \ trapped \ surface.$$

Example

(1) $\boldsymbol{n}^T \equiv \boldsymbol{v}$:*constant* $\Leftrightarrow M \subset H(\boldsymbol{v}:c)$: *a hypersurface in a spacelike hyperplane.*

- *Special case* :$\boldsymbol{n}^T(u) = \boldsymbol{e}_0 \Rightarrow H(\boldsymbol{e}_0, 0) = \mathbb{R}^n_0$: *Eudclidean space*
- $\boldsymbol{n}^S(u)$: *the ordinary unit normal in the Euclidean sence.*

$\widetilde{K}^{\pm}_{\ell}(u) = K(u)$: *The Gauss-Kronecker curvature,*

$\widetilde{H}^{\pm}_{\ell}(u) = \pm H(u)$: *The mean curvature.* $\Rightarrow H \equiv 0$: *Minimal surfaces*

(2) $\boldsymbol{n}^T(u) = \boldsymbol{X}(u) \Rightarrow M = \boldsymbol{X}(U) \subset H^n(-1)$: *a hypersurface in Hyperbolic space*

- $\widetilde{\mathbb{L}}^{\pm}(u) = \widetilde{\boldsymbol{X}(u) \pm \boldsymbol{n}^S}(u)$: *the hyperbolic Gauss map*

$\widetilde{K}^{\pm}_{\ell}(u) = \widetilde{K}^{\pm}_{h}(u)$: *The horospherical Gauss-Kronecker curvature,*

$\widetilde{H}^{\pm}_{\ell}(u) = \widetilde{H}^{\pm}_{h}(u)$: *The horospherical mean curvature.*

$\Rightarrow \widetilde{H}^{\pm}_{h} \equiv 0$: *CMC* ± 1 *surfaces*

(3) $\boldsymbol{n}^S \equiv \boldsymbol{v}$:*constant* $\Leftrightarrow M$:*a spacelike hypersurface in a timelike hyperplane*

$\Rightarrow \widetilde{H}^{\pm}_{\ell} \equiv 0$: *Maximal surfaces*

(4) $\boldsymbol{n}^S(u) = \boldsymbol{X}(u) \Rightarrow M = \boldsymbol{X}(U) \subset S^n_1$: *a spacelike hypersurface in de Sitter space*

- $\widetilde{\mathbb{L}}^{\pm}(u) = \boldsymbol{n}^T\widetilde{(u) \pm \boldsymbol{X}}(u)$: *the de Sitter horospherical Gauss map*

§3 Spacelike submanifolds with codimension two

- M : closed orientable $(n-1)$-manifold, $f : M \longrightarrow \mathbb{R}^{n+1}_1$: spacelike embedding.
- $\mathbb{R}^{n+1}_1$: time-oriented $\Rightarrow$ globally exists $n^T : M \longrightarrow H^n(-1)$:future directed timelike unit normal vector field along $f(M)$
- The *global ligthtcone Gauss map* :

$$\widetilde{\mathbb{L}}^{\pm} : M \longrightarrow S^{n-1}_{+} \;:\; \widetilde{\mathbb{L}}^{\pm}(p) = \widetilde{n^T(p) \pm n^S}(p).$$

- The *global normalized lightcone Gauss-Kronecker curvature function*:

$$\widetilde{\mathcal{K}}^{\pm}_{\ell}(p) = \det(-\pi^t \circ d\widetilde{\mathbb{L}}^{\pm}_p).$$

Theorem (The Gauss-Bonnet type theorem)

M : a closed orientable, spacelike submanifold of codimension two in $\mathbb{R}^{n+1}_1$. Suppose that n is odd. Then

$$\int_M \widetilde{\mathcal{K}}_{\ell} d\mathfrak{v}_M = \frac{1}{2}\gamma_{n-1}\chi(M),$$

$\chi(M)$: the Euler characteristic of M, $d\mathfrak{v}_M$: the volume form of M, γ_{n-1} : the volume of the unit $(n-1)$-sphere S^{n-1}.

§4 Spacelike submanifolds with general codimension

- $X : U \longrightarrow \mathbb{R}^{n+1}_1$: a spacelike embedding of codimension k ($U \subset \mathbb{R}^s$, $M = X(U)$)
- $N_p(M)$: the pseudo-normal space at $p = X(u)$, a k-dim Lorentz vector space.
- Two kinds of pseudo spheres:

$$\begin{aligned} N_p(M; -1) &= \{v \in N_p(M) \mid \langle v, v \rangle = -1\} \\ N_p(M; 1) &= \{v \in N_p(M) \mid \langle v, v \rangle = 1\} \end{aligned}$$

- Two unit normal sperical normal bundles over M:

$$N(M; -1) = \bigcup_{p\in M} N_p(M; -1) \text{ and } N(M; 1) = \bigcup_{p\in M} N_p(M; 1).$$

- Remark that $N_p(M; \pm 1)$ are non-compact$\Rightarrow$ we cannot integrate on the fiber.
- $^{\exists}$future directed unit timelike normal vector field $n^T(p) \in N_p(M; -1)$ (fix!!)
- $N_1(M)_p[n^T] = \{\xi \in N_p(M; 1) \mid \langle \xi, n^T(p) \rangle = 0\}$:$k-1$-spacelike normal sphere.
- $N_1(M)[n^T] = \bigcup_{p\in M} N_1(M)_p[n^T]$: *spacelike unit normal bundle w.r.t n^T*
- Remark that $N_1(M)_p[n^T]$ is compact$\Rightarrow$ we can integrate on the fiber.

• The ***lightcone Gauss map*** of $N_1(M)[n^T]$:

$$\widetilde{\mathbb{LG}}(n^T) : N_1(M)[n^T] \longrightarrow S^{n-1}_+ \ : \ \widetilde{\mathbb{LG}}(n^T)(p,\xi) = n^T\widetilde{(p)} + \xi$$

• $\Pi^t : \widetilde{\mathbb{LG}}(n^T)^* T\mathbb{R}^{n+1}_1 = TN_1(M)[n^T] \oplus \mathbb{R}^{k+1} \longrightarrow TN_1(M)[n^T]$: the projection.

• $\widetilde{S}(n^T)_{(p,\xi)} = -\Pi^t_{\mathbb{LG}\widetilde{(n^T)}(p,\xi)} \circ d_{(p,\xi)}\widetilde{\mathbb{LG}}(n^T) : T_{(p,\xi)}N_1(M)[n^T] \longrightarrow T_{(p,\xi)}N_1(M)[n^T]$

:the ***lightcone shape operator***.

• $\widetilde{K}_\ell(n^T)(p,\xi) = \det \widetilde{S}(n^T)_{(p,\xi)}$: the ***lightcone Lipschitz-Killing curvature*** of $N_1(M)[n^T]$ at (p,ξ)

• Remark: We can apply the theory of Lagarangian/Legendrian singularities to investigate local properties o the lightcone Lipschitz-Killing curvature. However, we do not mention these results here.

Theorem

$$(\widetilde{\mathbb{LG}}(n^T)^* d\mathfrak{v}_{S^{n-1}_+})_{(p,\xi)} = |\widetilde{K}_\ell(n^T)(p,\xi)| d\mathfrak{v}_{N_1(M)[n^T](p,\xi)},$$

where $d\mathfrak{v}_{N_1(M)[n^T]}$ *: the canonical volume form of* $N_1(M)[n^T]$*,* $d\mathfrak{v}_{S^{n-1}_+}$ *: the canonical volume form of* S^{n-1}_+.

• $^{\exists}$a differential form $d\sigma_{k-2}(n^T)$ of degree $k-2$ on $N_1(M)[n^T]$ s.t its restriction to a fiber is the volume element of the unit $k-2$-sphere and

$$d\mathfrak{v}_{N_1(M)[n^T]} = d\mathfrak{v}_M \wedge d\sigma_{k-2}(n^T).$$

Proposition (Uniqueness)

Let $n^T, \overline{n}^T$ be future directed unit timelike normal vector fields along M. Then we have

$$\int_{N_1(M)_p[n^T]} |\widetilde{K}_\ell(n^T)(p,\xi)| d\sigma_{k-2}(n^T) = \int_{N_1(M)_p[\overline{n}^T]} |\widetilde{K}_\ell(\overline{n}^T)(p,\overline{\xi})| d\sigma_{k-2}(\overline{n}^T).$$

• The ***total absolute lightcone curvature*** of M at p (well-defined):

$$K_\ell^*(p) = \int_{N_1(M)_p[n^T]} |\widetilde{K}_\ell(n^T)(p,\xi)| d\sigma_{k-2}(n^T).$$

§4 Spacelike submanifolds with general codimension

- $f : M \longrightarrow \mathbb{R}^{n+1}_1$ (M: s-dim closed orientable manifold): a spacelike immersion
- The ***total absolute lightcone curvature*** of M :

$$\tau_\ell(M, f) = \frac{1}{\gamma_{n-1}} \int_M K^*_\ell(p) d\mathfrak{v}_M = \frac{1}{\gamma_{n-1}} \int_{N_1(M)[n^T]} |\widetilde{K}_\ell(n^T)(p, \xi)| d\mathfrak{v}_{N_1(M)[n^T]},$$

where γ_{n-1} is the volume of the unit $n-1$-sphere S^{n-1}.

Theorem (The Chern-Lashof type theorem)

(1) $\tau_\ell(M, f) \geq \gamma(M) \geq 2,$
(2) *if* $\tau_\ell(M, f) < 3$, *then* M *is homeomorphic to the sphere* S^s,
where $\gamma(M)$ *is the Morse number of* M.

- Problem: Suppose $\tau_\ell(M, f) = 2$.
  What kind of immersed spheres in $\mathbb{R}^{n+1}_1$ we have ?
- This problem leads the notion of lightlike convexity and lightlike tightness.

§4 Spacelike submanifolds with codimension two, revisited

- If $s = n - 1$, then $N_1(M)[n^T]$ is a double covering of M.
- $^{\exists}\sigma(p) = (p, n^S(p))$: global section of $N_1(M)[n^T]$. $\Rightarrow K^*_\ell(p) = |\widetilde{K}^+_\ell(p)| + |\widetilde{K}^-_\ell(p)|$.
- The ***positive/or negative total absolute curvature*** of M:

$$\tau^{\pm}_\ell(M, f) = \frac{1}{\gamma_{n-1}} \int_M |\widetilde{K}^{\pm}_\ell| d\mathfrak{v}_M.$$

Theorem (The strong Chern-Lashof type theorem)

$$\tau^{\pm}_\ell(M, f) \geq 1.$$

- Remark that $^{\exists}M$ such that $\tau^+_\ell(M, f) \neq \tau^-_\ell(M, f)$.
- Independent lightlike vectors v_i $(i = 1, 2) \Rightarrow$ lightlike hyperplanes $HP(v_i : c_i)$.
- If $HP(v_1 : c_1) \cap HP(v_2 : c_2) \neq \emptyset$, then $HP(v_1 : c_1) \cup HP(v_2 : c_2)$ dvides $\mathbb{R}^{n+1}_1$.

into 4 regions. ; Two ***timelike regions*** and two ***spacelike regions***.

- $f(M)$ is ***lightlike convex*** if $f(M)$ lies entirely in one of the closed half-spacelike regions determined by the tangent lightlike hypersurfaces of $f(M)$ at any $p \in M$.

Theorem

$\tau^{\pm}_\ell(M, f) = 1 \Leftrightarrow$ *M is homeomorphic to S^{n-1} and $f(M)$ is lightlike convex.*

Thank you very much for your attention!

And

Happy birthday Reiko and Keizo!