On relation between Potentials for Pluriharmonic maps and Para-pluriharmonic maps

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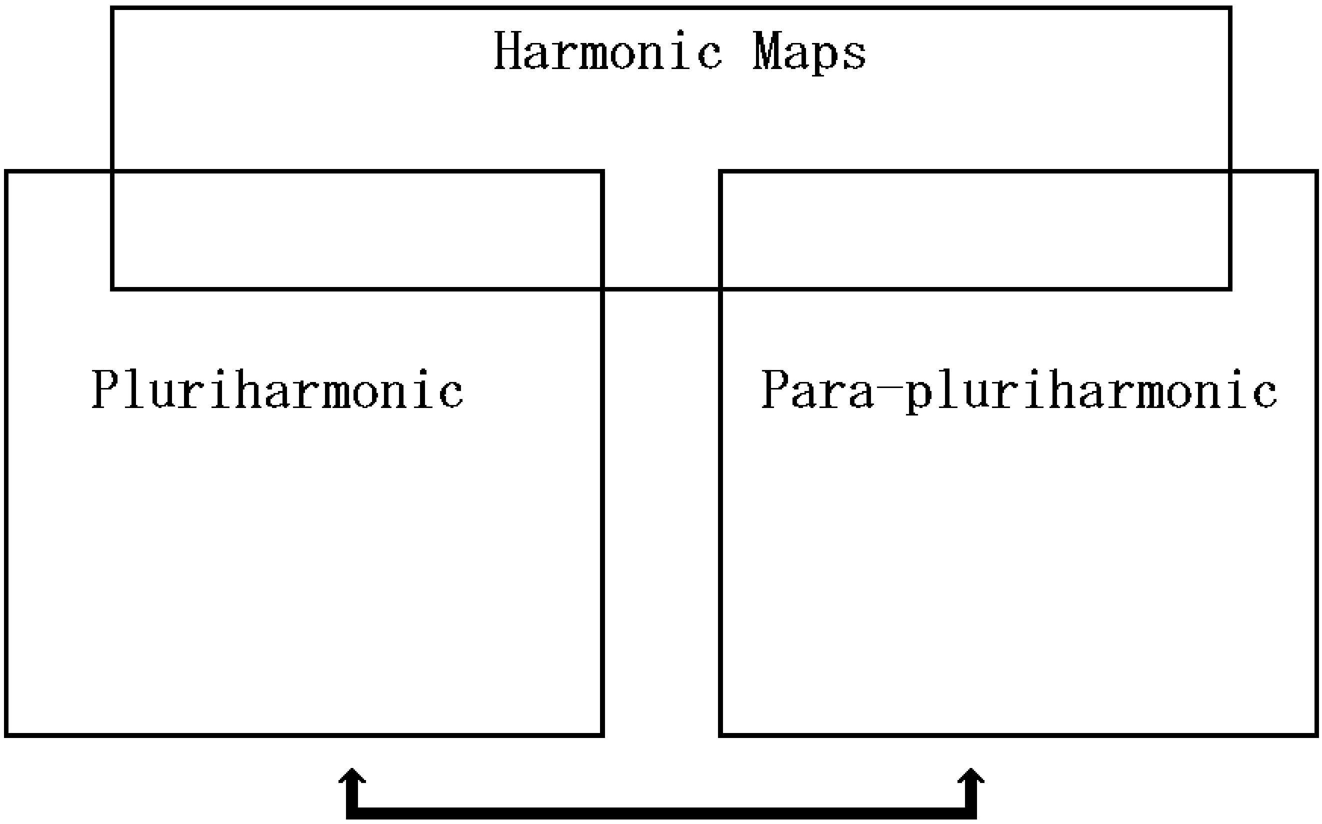
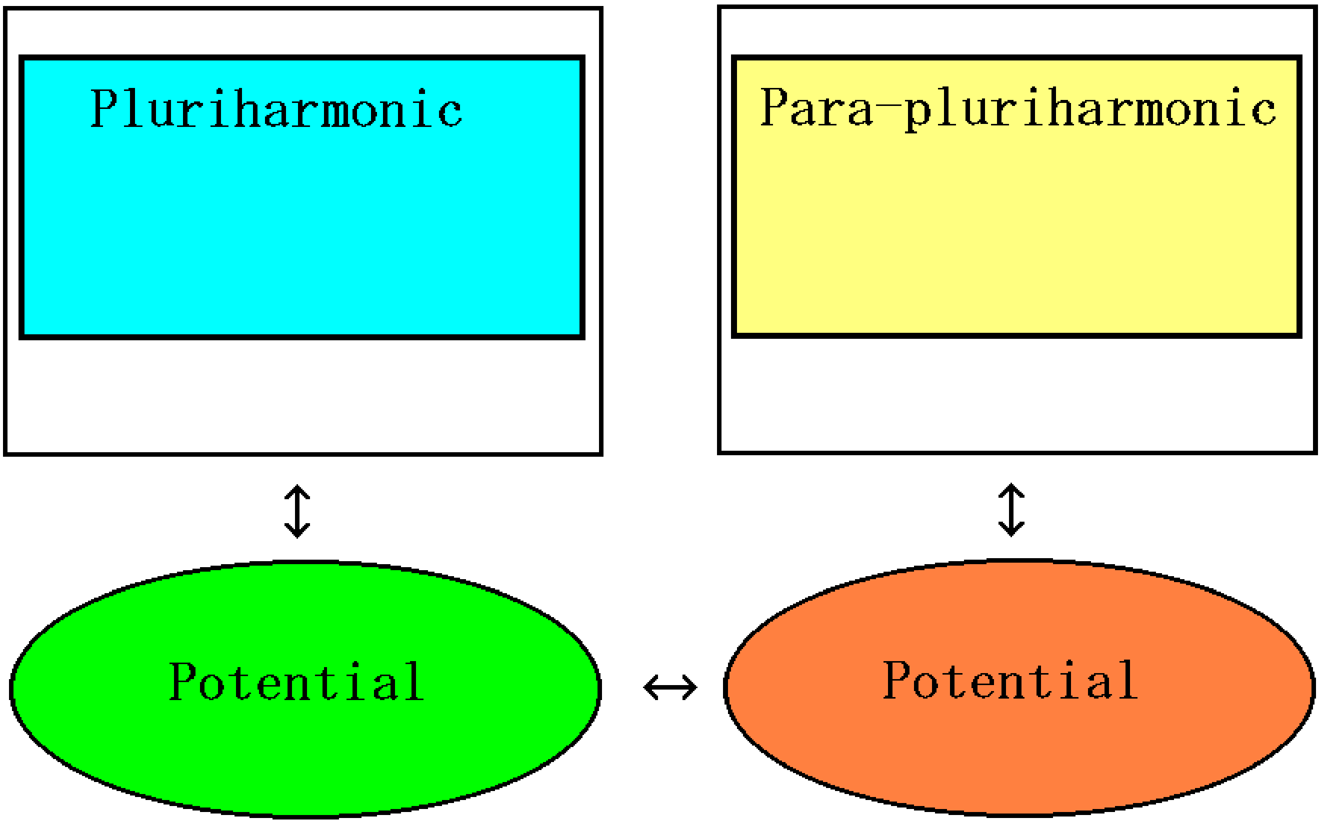
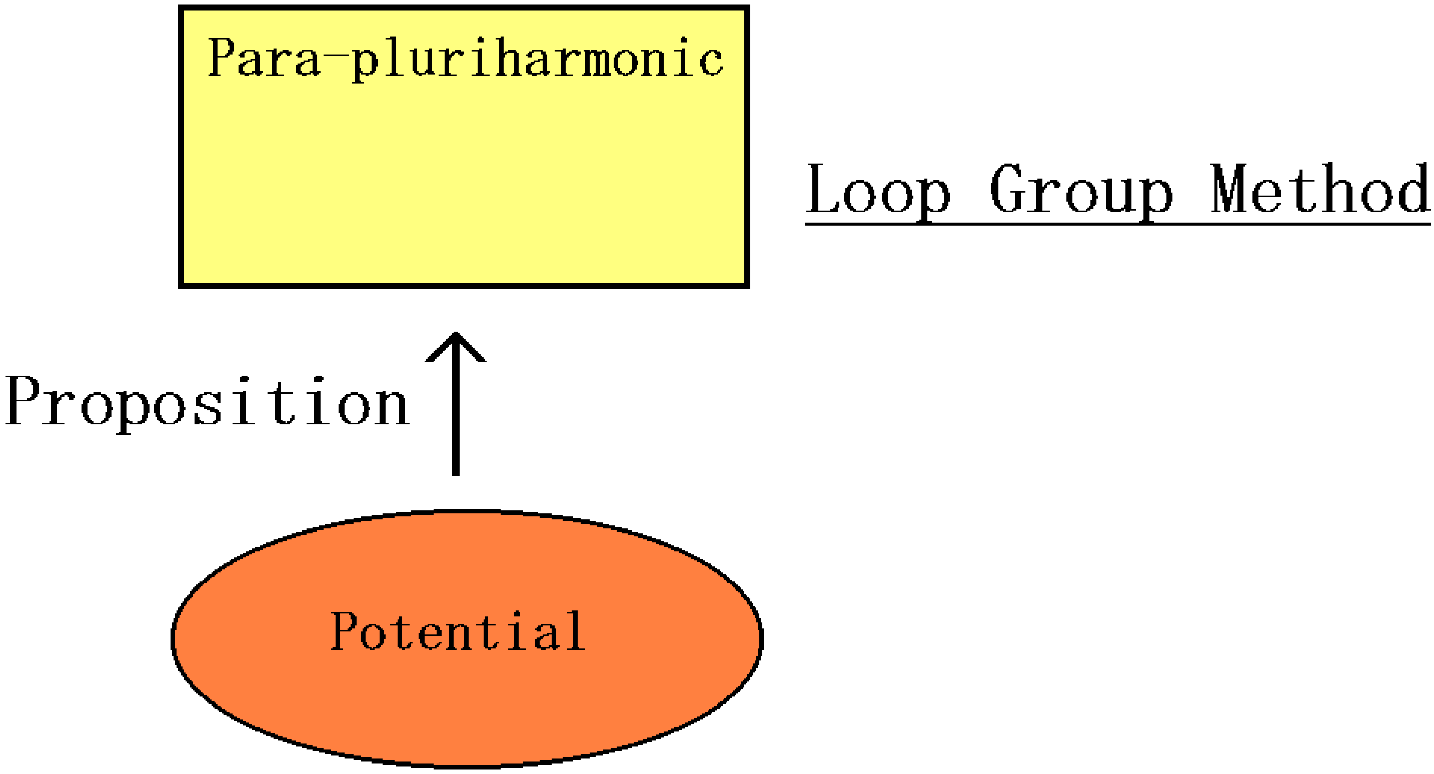
24 Jan. 2011 (Mon.): 14:00 – 14:50

※ joint work with **Josef Dorfmeister** (TU München)

1. Intro.




Pluriharmonic
Para-pluriharmonic
Potential
↔
Potential


Para-pluriharmonic
Loop Group Method
Proposition
Potential

Contents:

2. The Main Theorem

$$
\begin{array}{ccl}
f_1 : \mathbb{A}^{2n} & \longrightarrow & G_1/H_1, \text{ Pluriharmonic} \\
\cap & & \;\cap \\
\mathbb{C}^{2n} & & G^{\mathbb{C}}/H^{\mathbb{C}} \\
\cup & & \;\cup \\
f_2 : \mathbb{B}^{2n} & \longrightarrow & G_2/H_2, \text{ Para-pluriharmonic}
\end{array}
$$

2.1. **Subsp.** $\mathbb{A}^{2n}$ **and** $\mathbb{B}^{2n}$**:**

$$\mathbb{A}^{2n} := \{(z_i, w_i) \in \mathbb{C}^{2n} \mid w_i = \overline{z_i} \text{ for } {}^{\forall} i\},$$

$$\mathbb{B}^{2n} := \{(z_i, w_i) \in \mathbb{C}^{2n} \mid z_i = \overline{z_i}, w_i = \overline{w_i} \text{ for } {}^{\forall} i\}.$$

- Complex str. J on $\mathbb{A}^{2n}$:

$$J\Big(\frac{\partial}{\partial z_i}\Big) := \sqrt{-1}\frac{\partial}{\partial z_i},\ J\Big(\frac{\partial}{\partial \overline{z_i}}\Big) := -\sqrt{-1}\frac{\partial}{\partial \overline{z_i}}.$$

- Para-complex str. I on $\mathbb{B}^{2n}$:

$$I\Big(\frac{\partial}{\partial x_i}\Big) := \frac{\partial}{\partial x_i},\ I\Big(\frac{\partial}{\partial y_i}\Big) := -\frac{\partial}{\partial y_i},$$

where $x_i := \mathrm{Re}(z_i)$ and $y_i := \mathrm{Re}(w_i)$.

$\Longrightarrow$

- $(\mathbb{A}^{2n}, J)$ is a simply con. Complex mfd,
- $(\mathbb{B}^{2n}, I)$ is a simply con. Para-complex mfd.

$$
\begin{array}{ccc}
f_1 : (\mathbb{A}^{2n}, J) & \longrightarrow & G_1/H_1, \text{ Pluriharmonic} \\
\cap & & \cap \\
\mathbb{C}^{2n} & & G^{\mathbb{C}}/H^{\mathbb{C}} \\
\cup & & \cup \\
f_2 : (\mathbb{B}^{2n}, I) & \longrightarrow & G_2/H_2, \text{ Para-pluriharmonic}
\end{array}
$$

2.2. **Subsp.** G_1/H_1 **and** G_2/H_2**:**

$G^{\mathbb{C}}$: a simply con., Simple,

Complex linear alg. subgr. of $SL(m, \mathbb{C})$,

σ : a holo. involution of $G^{\mathbb{C}}$,

ν_a : an antiholo. involution of $G^{\mathbb{C}}$

s.t. $[\sigma, \nu_a] = 0$ and $[\nu_1, \nu_2] = 0$ $(a = 1, 2)$.

Notation.

(1) $H^{\mathbb{C}} := \mathrm{Fix}(G^{\mathbb{C}}, \sigma)$, (2) $G_a := \mathrm{Fix}(G^{\mathbb{C}}, \nu_a)$,

(3) $H_a := \mathrm{Fix}(G_a, \sigma) = \mathrm{Fix}(H^{\mathbb{C}}, \nu_a)$.

$\Longrightarrow$

- $(G^{\mathbb{C}}/H^{\mathbb{C}}, \sigma)$ is a Symm. sp.,
- G_a/H_a are Symm. subsp. of $(G^{\mathbb{C}}/H^{\mathbb{C}}, \sigma)$, $a = 1, 2$.

$$\begin{array}{ccccl} f_1 : (\mathbb{A}^{2n}, J) & \longrightarrow & (G_1/H_1, \nabla^1), & \text{Pluriharmonic} \\ \cap & & \cap & \\ \mathbb{C}^{2n} & & G^{\mathbb{C}}/H^{\mathbb{C}} & \\ \cup & & \cup & \\ f_2 : (\mathbb{B}^{2n}, I) & \longrightarrow & (G_2/H_2, \nabla^1), & \text{Para-pluriharmonic} \end{array}$$

Here, ∇^1 is the canonical aff. conn. on $(G_a/H_a, \sigma|_{G_a})$.

The Main Theorem (Boumuki-Dorfmeister)**.**

$(\eta_\theta(x_i), \tau_\theta(y_i)) \in \mathcal{P}_+(\mathfrak{g}_2) \times \mathcal{P}_-(\mathfrak{g}_2)$

: an analytic, para-pluriharm. Potential on $(\mathbb{B}^{2n}, I)$,

$(f_2)_\theta = \pi_2 \circ C_\theta(x_i, y_i) : (W, I) \longrightarrow (G_2/H_2, \nabla^1)$

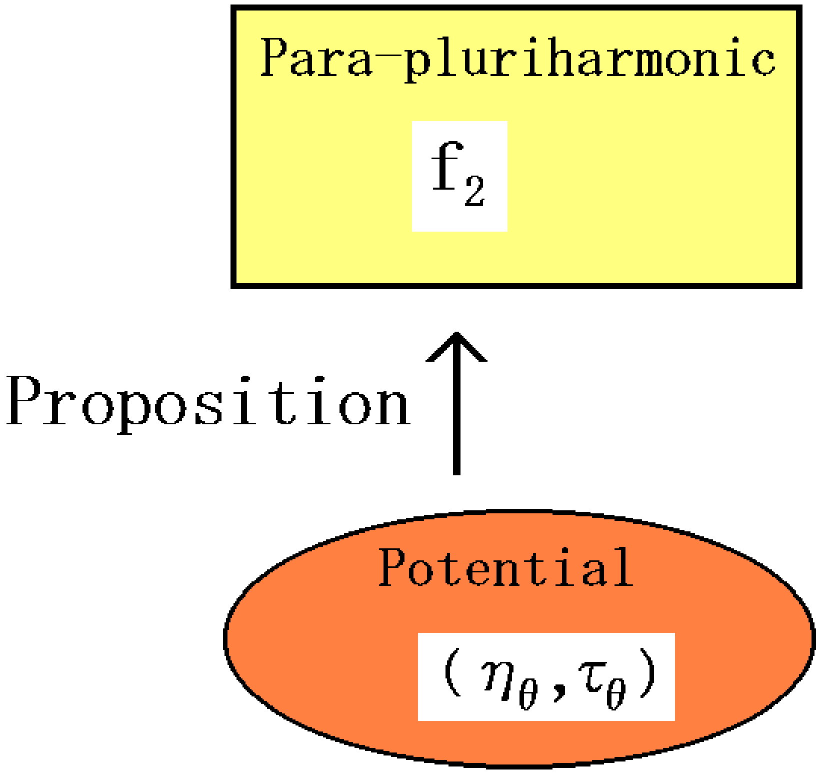
: the Para-pluriharm. map constructed from $(\eta_\theta, \tau_\theta)$

by **Proposition**.

※ W : an open neighbor. of $\mathbb{B}^{2n}$ at $(0, 0)$,

$C_\theta(x_i, y_i) : W \times \mathbb{R}^+ \to G_2$, map,

π_2 : the proj. from G_2 onto G_2/H_2.


Para-pluriharmonic
f2
Proposition
Potential
(ηθ , τθ)

Suppose: (M)

$$d(\nu_1)_C\big(\eta_\lambda(z_i)\big) = \tau_\lambda(\overline{z_i}) \text{ for } {}^\forall(z_i, \overline{z_i}; \lambda) \in \mathbb{A}^{2n} \times S^1.$$

$\Longrightarrow$

${}^\exists\, V$: an open neighbor. of $\mathbb{A}^{2n}$ at $(0,0)$,

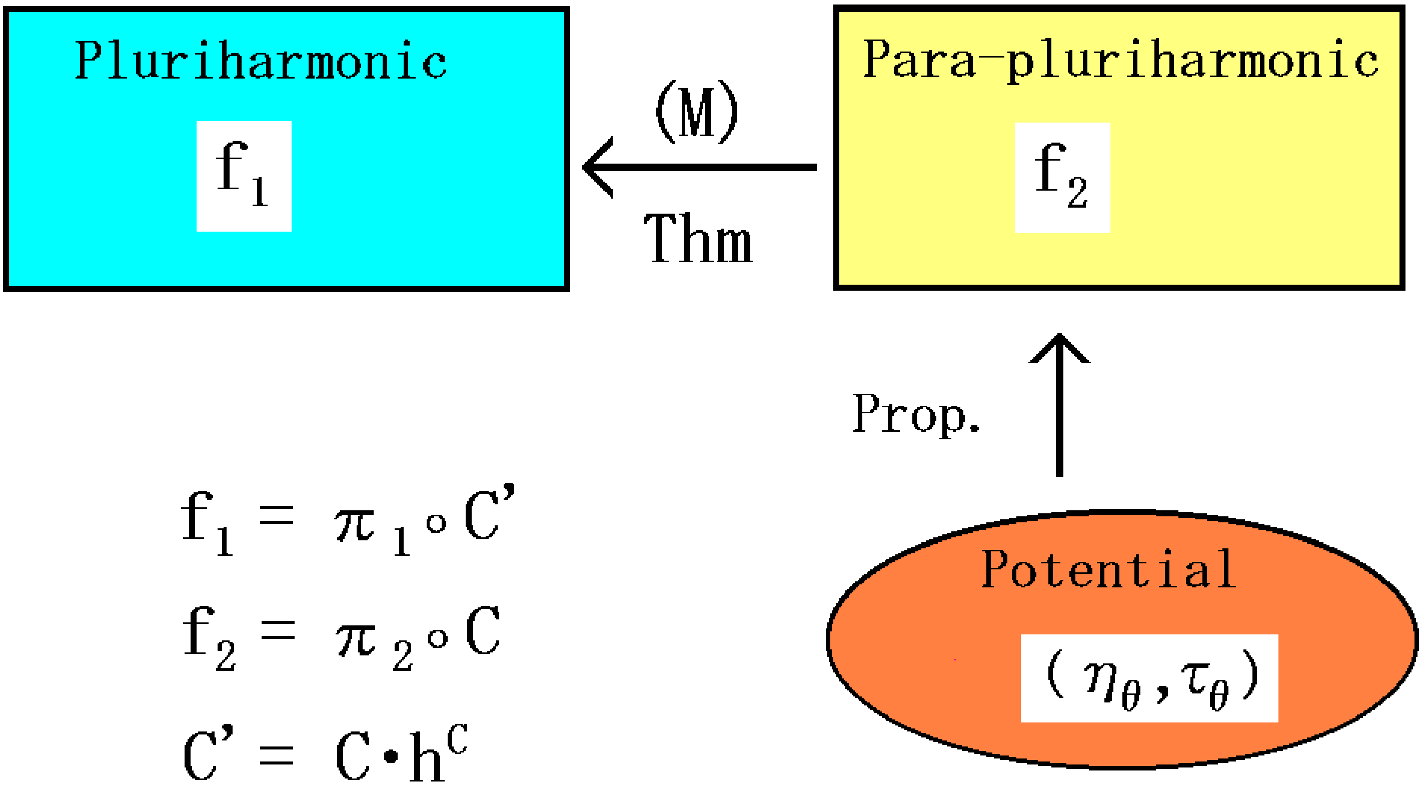
${}^\exists\, h^{\mathbb{C}}(z_i, \overline{z_i}) : V \longrightarrow H^{\mathbb{C}}$ s.t.

(1) $C'_\lambda(z_i, \overline{z_i}) \in G_1$ for ${}^\forall(z_i, \overline{z_i}; \lambda) \in V \times S^1$;

(2) $(f_1)_\lambda := \pi_1 \circ C'_\lambda(z_i, \overline{z_i}) : (V, J) \longrightarrow (G_1/H_1, \nabla^1)$

is a Pluriharm. map,

where $C'_\lambda(z_i, \overline{z_i}) := C_\lambda(z_i, \overline{z_i}) \cdot h^{\mathbb{C}}(z_i, \overline{z_i})$.



$$f_1 = \pi_1 \circ C'$$

$$f_2 = \pi_2 \circ C$$

$$C' = C \cdot h^c$$

The Main Theorem $\Longrightarrow$

From a para-pluriharm. **Potential** $(\eta_\theta, \tau_\theta)$ with **(M)**, we can obtain a **Pluriharm**. map $f_1 = \pi_1 \circ C'_\lambda : (V, J) \to (G_1/H_1, \nabla^1)$ and a **Para-pluriharm**. map $f_2 = \pi_2 \circ C_\theta : (W, I) \to (G_2/H_2, \nabla^1)$.

$$\begin{array}{ccc} f_1 : (V, J) & \longrightarrow & (G_1/H_1, \nabla^1), \text{ Pluriharmonic} \\ \cap & & \cap \\ \mathbb{C}^{2n} & & G^{\mathbb{C}}/H^{\mathbb{C}} \\ \cup & & \cup \\ f_2 : (W, I) & \longrightarrow & (G_2/H_2, \nabla^1), \text{ Para-pluriharmonic} \end{array}$$

※ $V \subset \mathbb{A}^{2n}$ & $W \subset \mathbb{B}^{2n}$, open.

Example.

$$
\begin{array}{llr}
 & & \text{Pluriharmonic} \\
f_1 : \mathbb{A}^2 & \longrightarrow SU(1,1)/S(U(1)\times U(1)) \simeq H^2 & \\
\quad\ \cap & \qquad\qquad\ \ \cap & \\
\quad\ \mathbb{C}^2 & \qquad SL(2,\mathbb{C})/S(GL(1,\mathbb{C})\times GL(1,\mathbb{C})) & \\
\quad\ \cup & \qquad\qquad\ \ \cup & \\
f_2 : \mathbb{B}^2 & \longrightarrow SL(2,\mathbb{R})/S(GL(1,\mathbb{R})\times GL(1,\mathbb{R})) \simeq S^2_1 & \\
 & & \text{Para-pluriharmonic}
\end{array}
$$

※ H^2 : an upper half-plane

S^2_1 : a hyperboloid of one sheet.

$$(\eta_\theta(x), \tau_\theta(y)) = \left(\begin{pmatrix} 0 & 1/\theta \\ 0 & 0 \end{pmatrix} dx, \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix} dy \right)$$

is an analytic, para-pluriharm. Potential on $(\mathbb{B}^2, I)$

satisfying **(M)**.

$\Longrightarrow$

$$(f_2)_\theta = \pi_2 \circ C_\theta(x, y) : (W, I) \longrightarrow G_2/H_2 = S^2_1,$$

$$C_\theta(x, y) = \frac{1}{\sqrt{1 - xy}} \begin{pmatrix} 1 & x/\theta \\ \theta y & 1 \end{pmatrix},$$

is the Para-pluriharm. map constructed from $(\eta_\theta, \tau_\theta)$

by **Proposition**,

where $W := \{(x, y) \in \mathbb{B}^2 \mid xy \neq 1\}$.

In this case,

$$V = \{(z, \overline{z}) \in \mathbb{A}^2 \,|\, z\overline{z} \neq 1\},\ h^{\mathbb{C}}(z, \overline{z}) = \mathrm{id}\,.$$

$$\Longrightarrow (f_1)_\lambda = \pi_1 \circ C'_\lambda(z, \overline{z}) : (V, J) \longrightarrow G_1/H_1 = H^2,$$

$$C'_\lambda(z, \overline{z}) = \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z/\lambda \\ \lambda\overline{z} & 1 \end{pmatrix},$$

is a Pluriharmon. map.

$f_1(z, \overline{z}) : (V, J) \longrightarrow H^2$, Pluriharmonic.

$$\Updownarrow$$

$f_2(x, y) : (W, I) \longrightarrow S^2_1$, Para-pluriharmonic.

$$f_1 = \pi_1 \circ C'_\lambda : V \to H^2, \qquad f_2 = \pi_2 \circ C_\theta : W \to S^2_1,$$

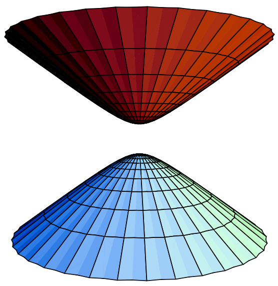
$$C'_\lambda(z, \overline{z}) = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z/\lambda \\ \lambda \overline{z} & 1 \end{pmatrix}, \; C_\theta(x, y) = \frac{1}{\sqrt{1-xy}} \begin{pmatrix} 1 & x/\theta \\ \theta y & 1 \end{pmatrix},$$

$$h^{\mathbb{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Digression (What is interesting ?):

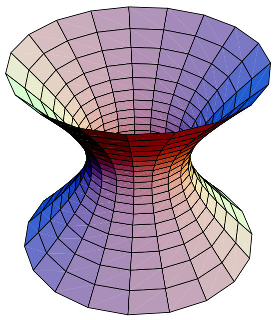
$f_1(z, \overline{z}) : V \longrightarrow H^2$

$\Longrightarrow$ a spacelike CMC-surface in $\mathbb{R}^3_1$



$f_2(x, y) : W \longrightarrow S^2_1$

$\Longrightarrow$ a timelike CMC-surface in $\mathbb{R}^3_1$



(e.g. $u_{z\bar{z}} = -\sinh u \longrightarrow$ CMC-surf. in $\mathbb{R}^3$)

Pluriharm. map $\longrightarrow$ elliptic PDE

$\Updownarrow$ $\Updownarrow$

Para-pluriharm. map $\longrightarrow$ hyperbolic PDE

(e.g. $u_{xy} = \sin u \longrightarrow$ K-surf. in $\mathbb{R}^3$)

3. Preliminaries (Definitions)

3.1. **Pluriharmonic & Para-pluriharmonic.**

Definition.

(M, g), (N, h) : pseudo-Riem. mfds,

where M is oriented.

A map $f : (M, g) \longrightarrow (N, h)$ is **Harmonic**, if

$$\text{Trace}_g(\overline{\nabla} df) = 0. \tag{1}$$

⋇ $\overline{\nabla}$ is the induced conn. on $\text{End}(TM, f^{-1}TN)$ from the Levi-Civita conn. on (M, g) and (N, h).

Remark. In case of $\dim_{\mathbb{R}} M = 2$,

- Definition of harm. map depends **Only** on the Conformal Class of g;
- the Conf. Class corresponds to the Complex Str. on M (up to sign) **when** g is a Riem. metric;
- the Conf. Class corresponds to the Para-complex Str. on M (up to sign) **when** g is a Lorentz. metric.

Therefore,

- $\mathrm{Trace}_g(\overline{\nabla} df) = 0 \iff (\overline{\nabla} df)(\partial/\partial\overline{z}, \partial/\partial z) = 0$

for ${}^\forall(z, \overline{z})$: local holo. coord. on M **when** (M, g) is a Riem. surf.

- $\mathrm{Trace}_g(\overline{\nabla} df) = 0 \iff (\overline{\nabla} df)(\partial/\partial y, \partial/\partial x) = 0$

for ${}^\forall(x, y)$: local para-holo. coord. on M **when** (M, g) is a Lorentz surf.

Definition 1.

(M, J) : a Complex mfd with $\dim_{\mathbb{R}} M = 2n$,

(N, ∇^N) : an Aff. mfd, where ∇^N is torsion-free.

A map $f : (M, J) \to (N, \nabla^N)$ is **Pluriharmonic**, if

(H) $$(\nabla df)\Big(\frac{\partial}{\partial \overline{z_a}}, \frac{\partial}{\partial z_b}\Big) = 0 \text{ for } 1 \leq {}^\forall a, b \leq n,$$

for ${}^\forall(z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n})$: local holo. coord. on (M, J).

※ ∇ denotes the induced conn. on $\mathrm{End}(TM, f^{-1}TN)$ from D and ∇^N, where D is any complex (i.e., $DJ = 0$) torsion-free aff. conn. on (M, J).

Definition 2 (Schäfer 2006)**.**

(M, I) : a Para-complex mfd with $\dim_{\mathbb{R}} M = 2n$,

(N, ∇^N) : an Aff. mfd, where ∇^N is torsion-free.

A map $f : (M, I) \to (N, \nabla^N)$ is **Para-pluriharmonic**, if

$$\text{(P)} \qquad (\nabla df)\Big(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial x_b}\Big) = 0 \text{ for } 1 \leq {}^{\forall}a, b \leq n,$$

for ${}^{\forall}(x_1, \ldots, x_n, y_1, \ldots, y_n)$: local para-holo. coord. on (M, I).

※ ∇ denotes the induced conn. on $\mathrm{End}(TM, f^{-1}TN)$ from D and ∇^N, where D is any para-complex torsion-free aff. conn. on (M, I).

Proposition (Ohnita-Valli 1990).

$$(M, J) \ : \ \text{a Complex mfd},$$

$$(N, h) \ : \ \text{a pseudo-Riem. mfd.}$$

A map $f : (M, J) \longrightarrow (N, \nabla^h)$ is Pluriharmonic

$$\Longleftrightarrow$$

$f \circ \imath : (\Sigma^2, g) \longrightarrow (N, h)$ is Harmonic for every holo. curve $\imath : (\Sigma^2, j) \to (M, J)$ from any Riem. surf. (Σ^2, j, g).

Here, ∇^h is the Levi-Civita conn. on (N, h).

Proposition (Schäfer 2006)**.**

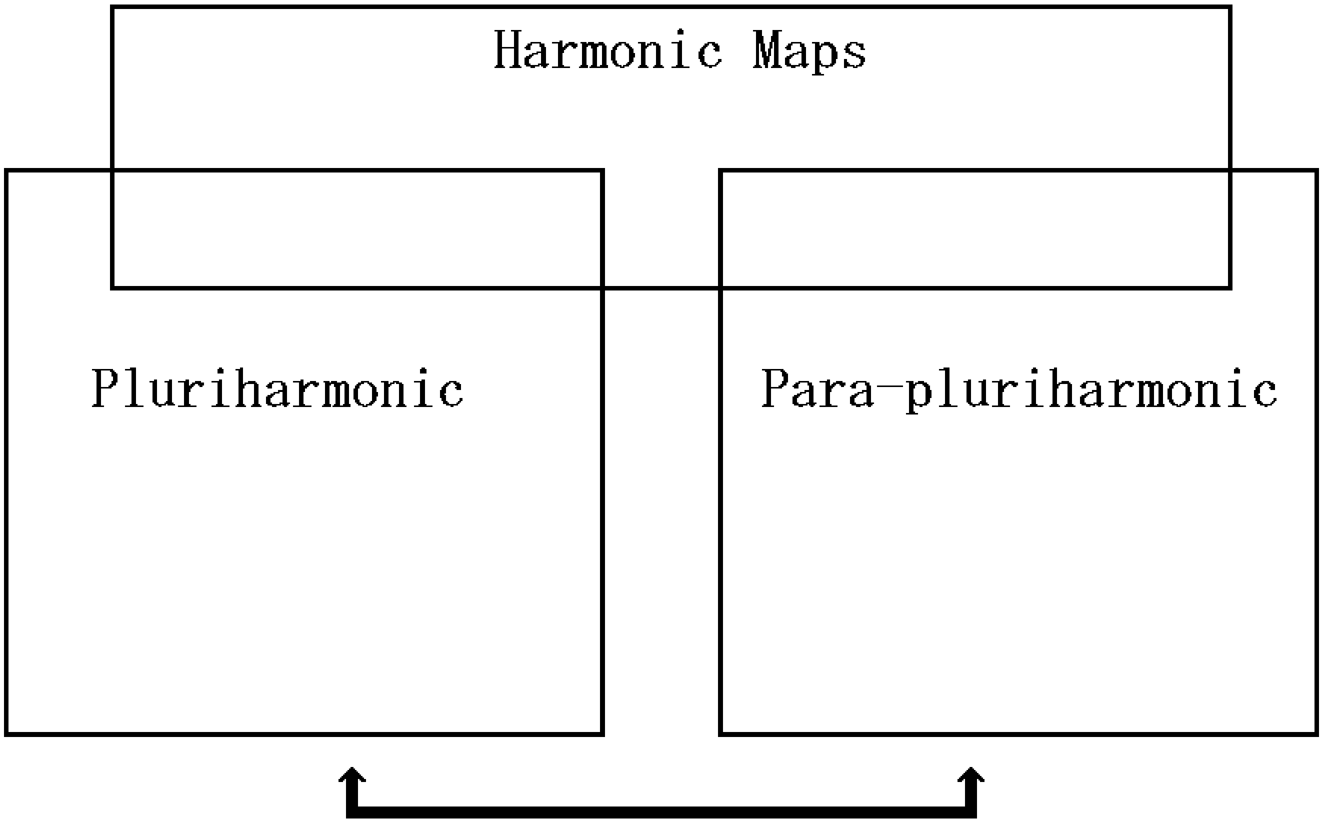
$$(M, I) \ : \text{ a Para-complex mfd,}$$

$$(N, h) \ : \text{ a pseudo-Riem. mfd.}$$

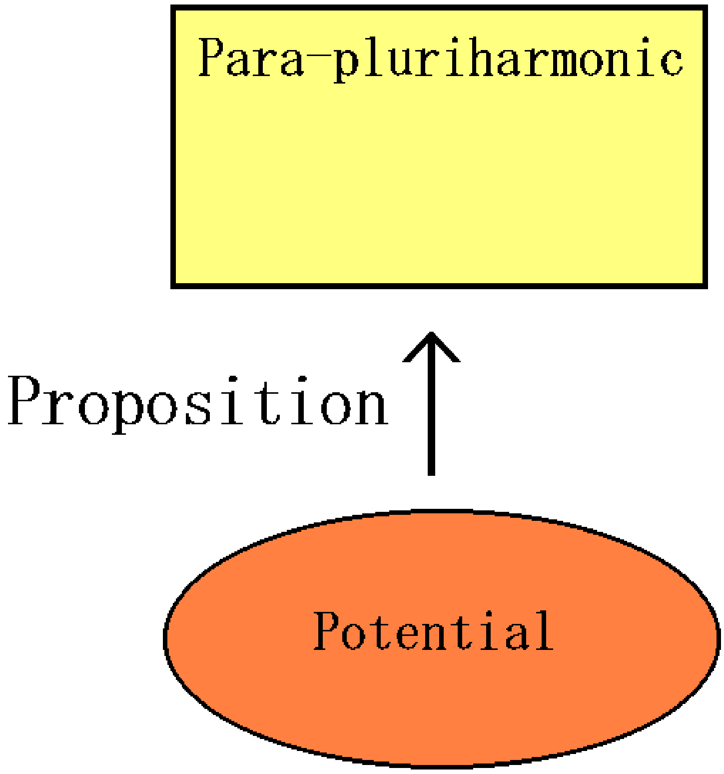
A map $f : (M, I) \longrightarrow (N, \nabla^h)$ is Para-pluriharmonic

$$\Longleftrightarrow$$

$f \circ \imath : (\Sigma_1^2, g) \longrightarrow (N, h)$ is Harmonic for every para-holo. curve $\imath : (\Sigma_1^2, i) \to (M, I)$ from any Lorentz surf. (Σ_1^2, i, g).


Harmonic Maps
Pluriharmonic
Para-pluriharmonic

Loop Group Method



3.2. **Loop group** $\Lambda G^{\mathbb{C}}_\sigma$.

$G^{\mathbb{C}}$: a simply con., Simple,

Complex linear alg. Subgr. of $SL(m, \mathbb{C})$,

σ : a holo. involution of $G^{\mathbb{C}}$,

ν : an antiholo. involution of $G^{\mathbb{C}}$ s.t. $[\sigma, \nu] = 0$.

$\Longrightarrow$ The **twisted loop group** $\Lambda G^{\mathbb{C}}_\sigma$ is given by

$$\Lambda G^{\mathbb{C}}_\sigma := \left\{ A_\lambda : S^1 \to G^{\mathbb{C}} \;\middle|\; \begin{array}{l} A_\lambda = \sum_{k \in \mathbb{Z}} A_k \lambda^k, \\ \sum_{k \in \mathbb{Z}} ||A_k|| < \infty, \\ \sigma(A_\lambda) = A_{-\lambda} \text{ for } {}^\forall \lambda \in S^1 \end{array} \right\},$$

where $|| \cdot ||$ is a matrix norm s.t. $||A \cdot B|| \leq ||A|| \cdot ||B||$

& $||\,\mathrm{id}\,|| = 1$.

Fact (Balan-Dorfmeister 2005; Goodman-Wallach 1984).

$\Lambda G_\sigma^{\mathbb{C}}$ is a complex Banach Lie gr.

w.r.t. $||A_\lambda|| := \sum_{k\in\mathbb{Z}} ||A_k||$.

3.3. Iwasawa Decomposition of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$.

- Define an antiholo. inv. ν_S of $\Lambda G_\sigma^{\mathbb{C}}$ by

$$\nu_S(A_\lambda) := \nu(A_{\overline{\lambda}}) \text{ for } A_\lambda \in \Lambda G_\sigma^{\mathbb{C}}.$$

Notation: $\Lambda G_\sigma := \mathrm{Fix}(\Lambda G_\sigma^{\mathbb{C}}, \nu_S)$.

• Define subgr. $\Lambda^{\pm}G_\sigma$ and $\Lambda^{\pm}_{*}G_\sigma$ of ΛG_σ by

$$\Lambda^{+}G_\sigma := \{A_\lambda \in \Lambda G_\sigma \,|\, A_\lambda = \textstyle\sum_{k\geq 0} A_k \lambda^k\},$$

$$\Lambda^{-}G_\sigma := \{A_\lambda \in \Lambda G_\sigma \,|\, A_\lambda = \textstyle\sum_{m\leq 0} A_m \lambda^m\},$$

$$\Lambda^{\pm}_{*}G_\sigma := \{A_\lambda \in \Lambda^{\pm} G_\sigma \,|\, A_0 = \mathrm{id}\}.$$

• Define a subgr. $\widetilde{\Lambda}G_\sigma$ of ΛG_σ by

$$\widetilde{\Lambda}G_\sigma := \left\{ A_\lambda \in \Lambda G_\sigma \,\middle|\, \begin{array}{r} A_\lambda \text{ has an analytic ext.} \\ \widetilde{A}_\mu : \mathbb{C}^* \to G^{\mathbb{C}} \end{array} \right\},$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Define subgr. $\widetilde{\Lambda}^{\pm}G_\sigma$ and $\widetilde{\Lambda}^{\pm}_{*}G_\sigma$ in a similar way.

Iwasawa decomposition of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$:

Theorem (Brander-Dorfmeister 2009;

Dorfmeister-Inoguchi-Toda 2002).

The multiplication map

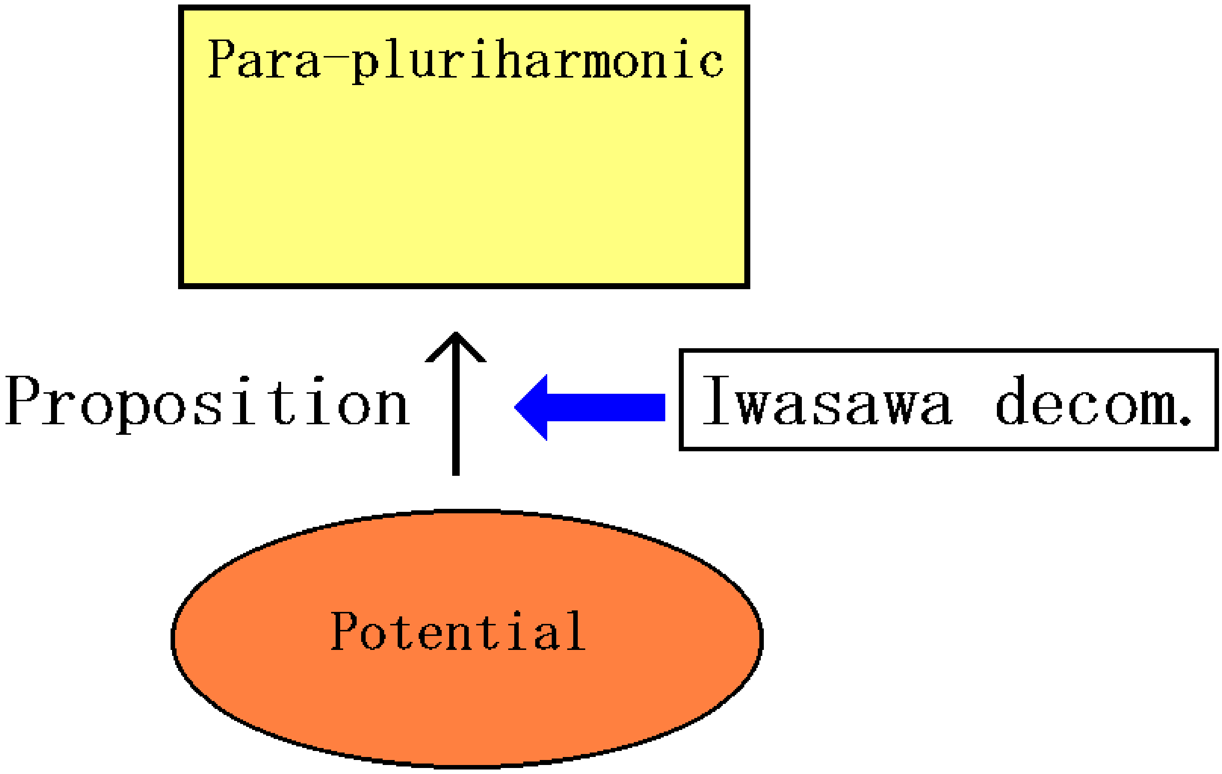
$$\triangle(\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma) \times (\widetilde{\Lambda}^+_* G_\sigma \times \widetilde{\Lambda}^- G_\sigma) \to \widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$$

$$\big((C_\lambda, C_\lambda), (B^+_\lambda, B^-_\lambda)\big) \mapsto (C_\lambda \cdot B^+_\lambda, C_\lambda \cdot B^-_\lambda)$$

is a real analytic diffeom. onto the open subset

of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$.

Here, $\triangle(\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma)$ denotes the diagonal subgr. of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$.


Para-pluriharmonic
Proposition
Iwasawa decom.
Potential

Remark.

We may consider that for any $A_\lambda \in \widetilde{\Lambda} G_\sigma$, the variable λ varies not only in S^1 but also in $\mathbb{R}^+$ (or more generally in $\mathbb{C}^*$). Because $A_\lambda : S^1 \to G^{\mathbb{C}}$ has an analytic ext. $\widetilde{A}_\mu : \mathbb{C}^* \to G^{\mathbb{C}}$.

Here $\mathbb{R}^+ := \{\theta \in \mathbb{R} \,|\, \theta > 0\}$.

4. Relation between Para-pluriharmonic maps and Potentials (**Proposition**)

4.1. **Potential.**

(1) $G^{\mathbb{C}}$: a simply con., Simple,

Complex linear alg. Subgr. of $SL(m, \mathbb{C})$,

(2) σ : a holo. inv. of $G^{\mathbb{C}}$,

(3) ν : an antiholo. inv. of $G^{\mathbb{C}}$ s.t. $[\sigma, \nu] = 0$.

(4) $H^{\mathbb{C}} := \mathrm{Fix}(G^{\mathbb{C}}, \sigma)$, (5) $G := \mathrm{Fix}(G^{\mathbb{C}}, \nu)$.

(6) $H := \mathrm{Fix}(G, \sigma) = \mathrm{Fix}(H^{\mathbb{C}}, \nu)$.

(7) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$: the canonical decom. of $(\mathfrak{g}, d\sigma)$,

where $\mathfrak{g} := \mathrm{Lie}\, G$.

(8) π : the proj. from G onto G/H.

Consider subsp. $\widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma$ and $\widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma$ of $\widetilde{\Lambda}\mathfrak{g}_\sigma$:

$$\widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma := \{X_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma \,|\, X_\lambda = \textstyle\sum_{i=-1}^{\infty} X_i \lambda^i\},$$

$$\widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma := \{Y_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma \,|\, Y_\lambda = \textstyle\sum_{j=-\infty}^{1} Y_j \lambda^j\},$$

where $\widetilde{\Lambda}\mathfrak{g}_\sigma$ is the Lie alg. of $\widetilde{\Lambda}G_\sigma$.

$$\mathcal{P}_+(\mathfrak{g}) := \left\{ \begin{array}{r} \widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma\text{-valued para-holo.} \\ \text{closed 1-forms on } (M, I) \end{array} \right\},$$

$$\mathcal{P}_-(\mathfrak{g}) := \left\{ \begin{array}{r} \widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma\text{-valued para-antiholo.} \\ \text{closed 1-forms on } (M, I) \end{array} \right\}.$$

Here, (M, I) is a simply con. Para-complex mfd.

Definition 3. $(\eta_\lambda, \tau_\lambda) \in \mathcal{P}_+(\mathfrak{g}) \times \mathcal{P}_-(\mathfrak{g})$ is a **para-pluriharmonic Potential** on (M, I).

Remark.

For each $(\eta_\lambda, \tau_\lambda) \in \mathcal{P}_+(\mathfrak{g}) \times \mathcal{P}_-(\mathfrak{g})$, the variable λ can vary not only in S^1 but also in $\mathbb{R}^+$ (or more generally in $\mathbb{C}^*$).

Proposition.

$(\eta_\theta, \tau_\theta) \in \mathcal{P}_+(\mathfrak{g}) \times \mathcal{P}_-(\mathfrak{g})$

: a para-pluriharm. Potential on (M, I).

Then, the following steps (S1), (S2) and (S3) provide an $\mathbb{R}^+$-family $\{f_\theta\}_{\theta \in \mathbb{R}^+}$ of Para-pluriharm. maps:

(S1) Solve the two initial value problems:

$$(A_\theta^-)^{-1} \cdot dA_\theta^- = \eta_\theta,\ (A_\theta^+)^{-1} \cdot dA_\theta^+ = \tau_\theta,$$

$$A_\theta^\pm(p_o) \equiv \mathrm{id},$$

where p_o is a base point in (M, I).

(S2) Factorize $(A^-_\theta, A^+_\theta) \in \widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$ in the Iwasawa decom.:

$$(A^-_\theta, A^+_\theta) = (C_\theta, C_\theta) \cdot (B^+_\theta, B^-_\theta),$$

where $C_\theta \in \widetilde{\Lambda} G_\sigma$, $B^+_\theta \in \widetilde{\Lambda}^+_* G_\sigma$ and $B^-_\theta \in \widetilde{\Lambda}^- G_\sigma$.

(S3) Then, $f_\theta := \pi \circ C_\theta : (W, I) \longrightarrow (G/H, \nabla^1)$ becomes a Para-pluriharm. map for ${}^\forall \theta \in \mathbb{R}^+$.

Here, W is any open neighbor. of M at p_o such that both (S1) and (S2) are solved on W.

Proposition $\Longrightarrow$

From a para-pluriharm. **Potential** $(\eta_\theta, \tau_\theta) \in \mathcal{P}_+(\mathfrak{g}) \times \mathcal{P}_-(\mathfrak{g})$ on (M, I), we can obtain a **Para-pluriharm.** map $f_\theta = \pi \circ C_\theta : (W, I) \longrightarrow (G/H, \nabla^1)$.

Here, $W \subset M$ open.

The Main Theorem.

$(\eta_\theta(x_i), \tau_\theta(y_i)) \in \mathcal{P}_+(\mathfrak{g}_2) \times \mathcal{P}_-(\mathfrak{g}_2)$

: an analytic, para-pluriharm. Potential on $(\mathbb{B}^{2n}, I)$,

$(f_2)_\theta = \pi_2 \circ C_\theta(x_i, y_i) : (W, I) \longrightarrow (G_2/H_2, \nabla^1)$

: the Para-pluriharm. map constructed from $(\eta_\theta, \tau_\theta)$

by **Proposition**.

Here, W is an open neighbor. of $\mathbb{B}^{2n}$ at $(0,0)$, and ∇^1 is the canonical aff. conn. on $(G_2/H_2, \sigma|_{G_2})$.

Suppose: (M)

$$d(\nu_1)_C\big(\eta_\lambda(z_i)\big) = \tau_\lambda(\overline{z_i}) \text{ for } {}^\forall(z_i, \overline{z_i}; \lambda) \in \mathbb{A}^{2n} \times S^1.$$

$\Longrightarrow$

$^\exists\, V$: an open neighbor. of $\mathbb{A}^{2n}$ at $(0,0)$,

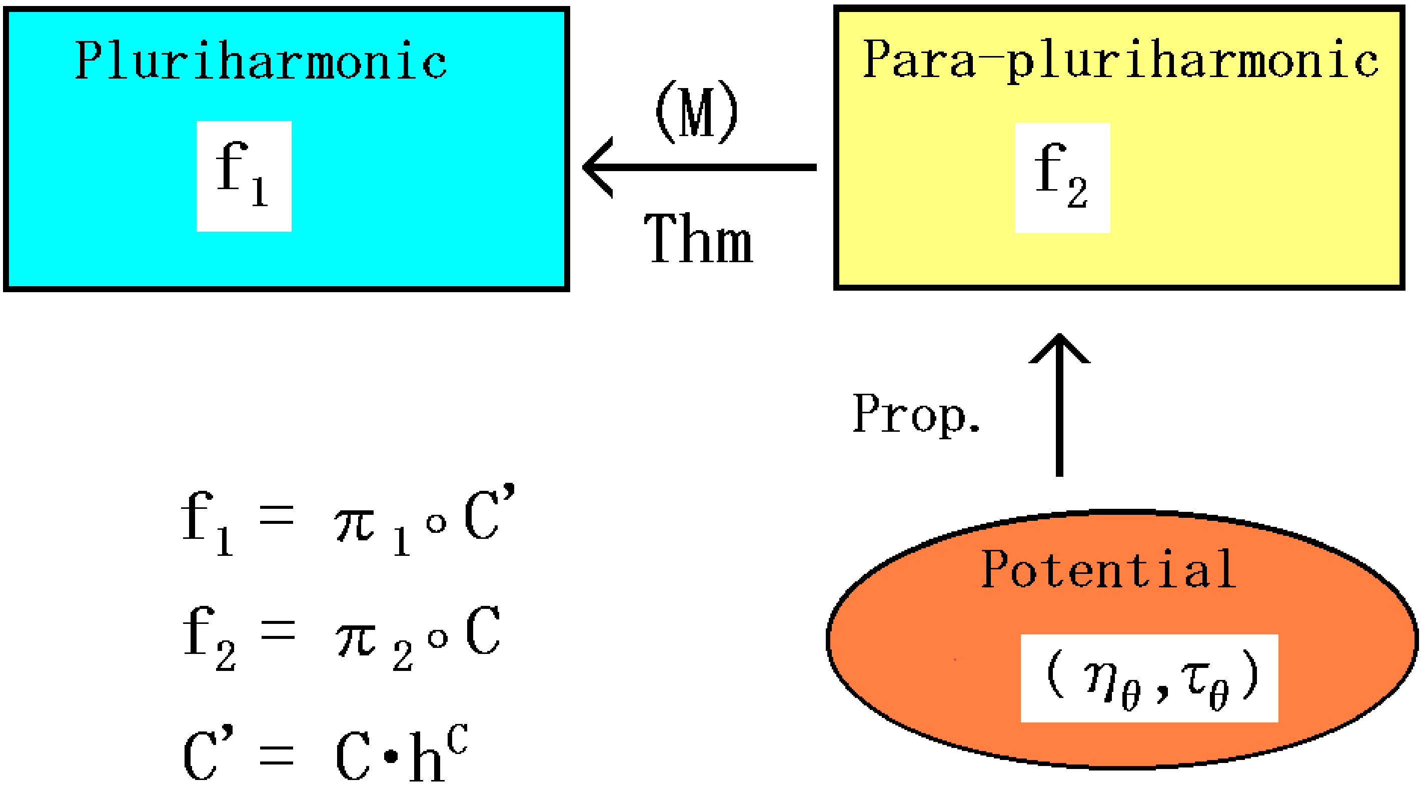
$^\exists\, h^{\mathbb{C}}(z_i, \overline{z_i}) : V \longrightarrow H^{\mathbb{C}}$ s.t.

(1) $C'_\lambda(z_i, \overline{z_i}) \in G_1$ for ${}^\forall(z_i, \overline{z_i}; \lambda) \in V \times S^1$;

(2) $(f_1)_\lambda := \pi_1 \circ C'_\lambda(z_i, \overline{z_i}) : (V, J) \longrightarrow (G_1/H_1, \nabla^1)$ is a Pluriharm. map,

where $C'_\lambda(z_i, \overline{z_i}) := C_\lambda(z_i, \overline{z_i}) \cdot h^{\mathbb{C}}(z_i, \overline{z_i})$.

Here, $(\nu_1)_C(A_\lambda) := \nu_1(A_{1/\overline{\lambda}})$ for $A_\lambda \in \Lambda G^{\mathbb{C}}_\sigma$.



$$f_1 = \pi_1 \circ C'$$

$$f_2 = \pi_2 \circ C$$

$$C' = C \cdot h^C$$

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Definition (Libermann 1952)**.**

$$M \ : \text{ a real } 2n\text{-dim. mfd.}$$

Then M is **Para-complex**, if ${}^{\exists}I$: a $(1,1)$-tensor field on M s.t.

(1) $I^2 = \mathrm{id}$;

(2) $\dim_{\mathbb{R}} T_p^{+}M = n = \dim_{\mathbb{R}} T_p^{-}M$ for ${}^{\forall}p \in M$;

(3) $[IX, IY] - I[IX, Y] - I[X, IY] + [X, Y] = 0$

for ${}^{\forall}X, Y \in \mathfrak{X}M$,

where $T_p^{\pm}M$ is the $\pm$-eigensp. of I_p (= the value of I at p) in T_pM.