Second variation formula and the stabilities of Legendrian minimal submanifolds in Sasakian manifolds

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Introduction

First, we consider the simplest example.

- (S^2, ω_0) : the sphere endowed with the canonical symplectic form.
- L^1 : the equator in S^2
  (Remark that all of 1-dim submanifolds in S^2 are Lagrangian submanifolds).

Deform L and consider volume variational problem.
$\rightsquigarrow$ L is not stable under general deformations.

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$$\text{"}\textit{area bisecting}\text{"} \iff V \rfloor \omega_0 \textit{ is an exact form.}$$

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We call such a special type of deformation Hamiltonian deformation.

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We call such a special type of deformation Hamiltonian deformation.

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In the general situation, we define the Hamiltonian deformation as follows.

Definition

(P^{2n}, ω), Symplectic manifold.
$\iota : L^n \to P^{2n}$, Lagrangian immersion, *i.e.*, $\iota^*\omega = 0$.
$\iota_t : L^n \to P^{2n}$, smooth deformation with variational vector field V.

$$\{\iota_t\} \text{ is a } \textit{Lagrangian deformation}. :\Longleftrightarrow \alpha_V := \iota^*(V\rfloor\omega) \text{ is } \textit{closed}.$$
$$\{\iota_t\} \text{ is a } \textit{Hamiltonian deformation}. :\Longleftrightarrow \alpha_V = \iota^*(V\rfloor\omega) \text{ is } \textit{exact}.$$

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H-minimal Lagrangian immersion

The variational problem of Lagrangian submanifolds under Hamiltonian deformations was first investigated by Y.G.Oh ([Oh1], [Oh2]).

Oh introduced the notion of *Hamiltonian-minimal* Lagrangian submanifolds in Kähler manifolds.

Definition

Let (P^{2n}, ω) be a Kähler manifold.
A Lagrangian immersion $\iota : L^n \to P^{2n}$ is called a *Hamiltonian-minimal* (*H-minimal*) if

$$\frac{d}{dt}\Big|_{t=0} Vol(\iota_t(L)) = 0,$$

for all Hamiltonian deformations $\{\iota_t\}$.

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In the following, we assume (P^{2n}, ω) be a Kähler manifold.

We derive the Euler-Lagrange equation of H-minimal Lagrangian immersions.

Theorem (Euler-Lagrange equation)

A Lagrangian immersion $L^n \to P^{2n}$ is H-minimal.
$\iff \delta\alpha_H = 0$,
where H is the mean curvature vector of L, $\alpha_H := \iota^*(H \rfloor \omega)$.

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Examples of H-minimal Lagrangian submanifolds

Examples. (1) Minimal ($i.e., H = 0$) $\Longrightarrow$ H-minimal.
(2) The standard tori in $\mathbb{C}^n$,

$$T^n = S^1(r_1) \times \cdots \times S^1(r_n)$$

are H-minimal Lagrangian but not minimal.

Second variation formula

We derive the second variation formula for H-minimal Lagrangian immersions ([Oh2]).

Theorem (Oh, 1993)

(P^{2n}, ω, J), Kähler manifold.
$\iota : L^n \longrightarrow P^{2n}$, H-minimal Lagrangian immersion.
If $\{\iota_t\}$ is a Hamiltonian deformation of L such that the variational vector field V is normal to L, then we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \{g(\Delta\alpha_V, \alpha_V) - \overline{\mathrm{Ric}}(V, V) - 2g(H, B(JV, JV)) + g(H, V)^2\}.$$

Hamiltonian stability

Definition

A H-minimal Lagrangian immersion $L^n \to P^{2n}$ is called Hamiltonian stable if

$$\frac{d^2}{dt^2}\Big|_{t=0} Vol(\iota_t(L)) \geq 0$$

for all Hamiltonian deformations $\{\iota_t\}$.

Example. The standard tori T^n in $\mathbb{C}^n$ are all Hamiltonian stable.

Legendrian immersion and Sasakian manifold

On the other hand, there is the notion of contact manifolds which is an odd-dimentional counterpart of symplectic manifolds.

Let (M^{2n+1}, η) be a contact manifold.
An immersion $\iota : L^n \rightarrow M^{2n+1}$ is called Legendrian if $\iota^* \eta = 0$.

In contact geometry, there are Sasakian manifolds which can be viewed as odd-dimentioal version of Kähler manifolds.

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In the following, we assume

- $(M^{2n+1}, \phi, \xi, \eta, g)$, Sasakian manifold.
- $\iota : L^n \to M^{2n+1}$, Legendrian immersion, *i.e.*, $\iota^*\eta = 0$.
- $\iota_t : L^n \to M^{2n+1}$, smooth deformation with $\iota_0 = \iota$.

Definition

$\{\iota_t\}$ is called Legendrian if ι_t is Legendrian immersion for each t (i.e., ι_t leave Legendrian submanifolds Legendrian).

Proposition

$\{\iota_t\}$ is Legendrian deformation.
$\Longleftrightarrow \iota^*(V \rfloor d\eta) = -d(\eta(V))$, *where V is the variational vector field.*

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L-minimal Legendrian immersion

Definition

A Legendrian immersion $\iota : L^n \to M^{2n+1}$ is called Legendrian minimal (L-minimal) if

$$\frac{d}{dt}\bigg|_{t=0} Vol(\iota_t(L)) = 0.$$

for all Legendrian deformations $\{\iota_t\}$.

Theorem (Euler-Lagrange equation)

$\iota : L^n \to M^{2n+1}$ is L-minimal.
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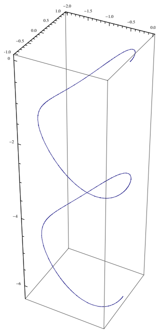
Examples of L-minimal Legendrian submanifolds

Examples. (1) L-minimal Legendrian curves in $(\mathbb{R}^3, \widetilde{g})$.

$$\gamma(s) = \Big(\frac{2}{h}\cos hs - \frac{2}{h}, \frac{2}{h}\sin hs, -\frac{2}{h}s + \frac{1}{h^2}\sin 2hs\Big)\ (h \neq 0).$$



case of h=2.

Examples of L-minimal Legendrian submanifolds

(2) All of L-minimal Legendrian closed curves in $S^3(1)$ is given by as follows ([SW], [Iriyeh]).

$$\gamma(s) = \frac{1}{\sqrt{p+q}}\left(\sqrt{q}e^{\sqrt{-1}\sqrt{\frac{p}{q}}s}, \sqrt{-1}\sqrt{p}e^{-\sqrt{-1}\sqrt{\frac{q}{p}}s}\right), \ 0 \leq s \leq 2\pi\sqrt{pq},$$

where (p,q) is a pair of relatively prime positive integers.
They are torus knots of type (p,q).

(3) L-minimal Legendrian flat tori $T^n_{(p_1,\cdots,p_{n+1})}$ in $S^{2n+1}(1)$, these are the generization of (2) ([Iriyeh]).

Main result

We derive the second variation formula for L-minimal Legendrian immersions.

Theorem (K)

$(M^{2n+1}, \phi, \xi, \eta, g)$, Sasakian manifold.
$\iota : L^n \to M^{2n+1}$, L-minimal Legendrian immersion.
If $\{\iota_t\}$ is a Legendrian deformation of L such that the variational vector field V is normal to L, denote $V = V_{\mathcal{H}} + f\xi$, then we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \{g(\Delta\alpha_V, \alpha_V) - 2g(\alpha_V, \alpha_V) - \overline{\mathrm{Ric}}(V_{\mathcal{H}}, V_{\mathcal{H}}) - 2g(H, B(\phi V_{\mathcal{H}}, \phi V_{\mathcal{H}})) + g(H, V_{\mathcal{H}})^2\},$$

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Legendrian stability

Definition

L-minimal Legendrian immersion $L^n \to M^{2n+1}$ is called Legendrian stable if

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(\iota_t(L)) \geq 0,$$

for all Legendrian deformations $\{\iota_t\}$.

Applications

We apply the second variation formula to study the stability of L-minimal Legendrian submanifolds.
First we prepare some basic curvature properties.

- A Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called η-Einstein Sasakian manifold if there exist constant A such that

$$\overline{\mathrm{Ric}} = Ag + (2n - A)\eta \otimes \eta.$$

- A Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called Sasakian space form if ϕ-sectional curvature is constant $(= c)$. We denote Sasakian space form $M(c)$.
- Sasakian space forms are η-Einstein with constant $A = \frac{n(c+3)+c-1}{2}$.

Example. Both $\mathbb{R}^{2n+1}(-3)$ and $S^{2n+1}(1)$ are Sasakian space form.

Applications

For L-minimal Legendrian cuves in 3-dim η-Einstein Sasakian manifolds, we have the following corollary from the second variation formula.

Corollary

$(M^3, \phi, \xi, \eta, g)$, 3-dim η-Einstein Sasakian manifold.
$L^1 \to M^3$, cpt L-minimal Legendrian curves.
Then

$$L^1 \text{ is Legenderian stable.} \iff \lambda_1 \geq A + 2 + h^2,$$

where $h^2 = |H|^2$, λ_1 is the first eigen value of Laplace-Beltrami operator Δ acting on $C^\infty(L)$.

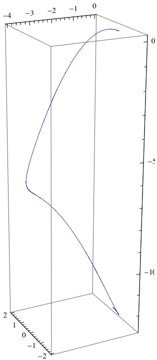
Moreover, $M^3(c)$ is Sasakian space form, then

$$L^1 \text{ is Legendrian stable.} \iff \lambda_1 \geq c + 3 + h^2.$$

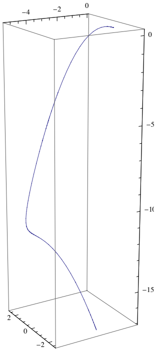
Examples. (1) L-minimal Legendrian curves in $\mathbb{R}^3(-3)$ with the length l are Legendrian stable iff

$$-\frac{\pi}{l} \leq h \leq \frac{\pi}{l}.$$

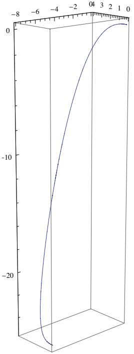
The case of $l = 2\pi$.



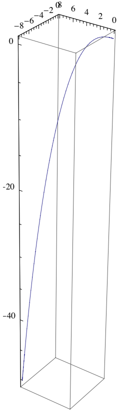
$h = 1$, unstable



$h = \frac{3}{4}$, unstable



$h = \frac{1}{2}$, stable



$h = \frac{1}{4}$, stable

Applications

(2) All of L-minimal Legendrian closed curves in $S^3(1)$ are unstable.

(3) L-minimal Legendrian tori $T^2_{(1,1,u)}$ in $S^5(1)$ are unstable.

Remark. It is already known that all of *minimal* cpt Legendrian submanifolds in S^{2n+1} are unstable ([H.Ono]).
Since this and above results, I conjecture all of L-minimal Legendrian tori $T^n_{(p_1,\cdots,p_{n+1})}$ in $S^{2n+1}(1)$ are unstable.

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References

Iriyeh H. Iriyeh, *Hamiltonian Minimal Lagrangian Cone in* $\mathbb{C}^m$, Tokyo J. Math. , Vol. 28, No. 1(2005), 91-107

Oh1 Y. G. Oh, *Second variation and stability of minimal Lagrangian submanifolds*, Invent. Math. , 101(1990), 501-519

Oh2 Y. G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Zeit. , 212(1993), 175-192

Ono H. Ono, *Second variation and Legendrian stabilities of minimal Legendrian submanifolds in Sasakian manifolds*, Differential Geometry and its Applications. , 22(2005), 327-340

SW R. Schoen, J. Wolfson, *Minimizing area among Lagrangian surfaces: The mapping problem*, J. Differential geometry. , 58(2001) 1-86

Thank you for your attention!