Rank 2 prolongations of second order PDE and geometric singular solutions

joint work with K. Shibuya (Hiroshima Univ)

Takahiro Noda

Nagoya Univ/OCAMI

2011/1/25.

Today's thema

1. Rank 2 prolongations of 2nd order single PDE .

2. Geometric singular solutions

Introduction

Let $J^2(\mathbb{R}^2, \mathbb{R})$ be the 2-jet space:

$$J^2(\mathbb{R}^2, \mathbb{R}) := \{(x, y, z, p, q, r, s, t)\}\,. \qquad (1)$$

This space has the canonical differential system (or higher order contact system)

$C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ given by the following 1-forms:

$$\varpi_0 := dz - pdx - qdy,$$

$$\varpi_1 := dp - rdx - sdy,$$

$$\varpi_2 := dq - sdx - tdy.$$

On $J^2(\mathbb{R}^2, \mathbb{R})$, we consider PDEs of the form:

$$F(x, y, z, p, q, r, s, t) = 0, \tag{2}$$

where $F \in C^\infty(J^2(\mathbb{R}^2, \mathbb{R}))$. We set

$$R := \{F = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R}), \;\; D := C^2|_R.$$

If we assume the regularity condition:

$$(F_r, F_s, F_t) \neq (0, 0, 0) \tag{3}$$

then,

(i) R is a smooth hypersurface,

(ii) the restriction $\pi^2_1|_R : R \to J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi^2_1 : J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$ is a submersion.

Hence, we have the induced differential system $D = \{\varpi_0|_R = \varpi_1|_R = \varpi_2|_R = 0\}$ on R.

Definition 1 Let (R, D) be a differential system given by $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$. Then, the rank 2 prolongation of (R, D) is defined by

$$\Sigma(R) := \bigcup_{x \in R} \Sigma_x, \tag{4}$$

where

$$\Sigma_x = \{v \subset D(x) | \ v \text{ is a 2-dim. integ. elem. of } D(x)\} .$$

(Integral element v is defined by $d\varpi_i|_v = 0$).

Let $p : \Sigma(R) \to R$ be the projection. If $\Sigma(R)$ is smooth, we define the canonical system $\hat{D}$ on $\Sigma(R)$ by

$$\begin{aligned} \hat{D}(u) &: = p_*^{-1}(u), \\ &= \{v \in T_u(\Sigma(R)) \mid p_*(v) \in u\} , \end{aligned} \tag{5}$$

where $u \in \Sigma(R)$.

This space $\Sigma(R)$ is a subset of the Grassmann bundle over R

$$J(D, 2) := \bigcup_{x \in R} J_x \tag{6}$$

where

$$J_x := \{v \subset D(x) \mid v \text{ is a 2-dim. subspace of } D(x)\}\,.$$

In general, rank 2 prolongations $\Sigma(R)$ have singular points!!

Rank 2 prolongations of regular PDEs

Here, we show that types of equations are characterized by the fiber topology of rank 2 prolongations of equations. To this purpose, we define

Definition 2

$R = \{F = 0\}$: 2nd order regular PDE.

For the discriminant of F:

$$\Delta := F_r F_t - \frac{1}{4}{F_s}^2,$$

a point $w \in R$ is said to be hyperbolic, parabolic or elliptic if $\Delta(w) < 0$, $\Delta(w) = 0$ or $\Delta(w) > 0$, respectively.

Rank 2 prolongations of hyperbolic equations

(R, D): locally hyperbolic. Then, $\exists$ a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{11}, \pi_{22}\}$ around $x \in R$ s.t.

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} \tag{7}$$

and

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \qquad \text{mod} \quad \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \pi_{11} \qquad \text{mod} \quad \varpi_0, \varpi_1, \varpi_2 \\ d\varpi_2 &\equiv \omega_2 \wedge \pi_{22} \qquad \text{mod} \quad \varpi_0, \varpi_1, \varpi_2 \end{aligned} \tag{8}$$

Theorem 3

Let (R, D) be a locally hyperbolic equation. Then, the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D, 2)$, and it is a T^2-bundle over R.

Remark 1 In fact, this result (i.e. $\Sigma(R)$ is torus bundle) is known by Bryant, Griffiths and Hsu by using the theory of the hyperbolic exterior differential system.

Outline of proof

We use the covering of $\pi : J(D,2) \to R$. For $U \subset R$ (open), $\pi^{-1}(U)$ is covered as follows:

$$\pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{11}} \cup U_{\omega_1\pi_{22}} \cup U_{\omega_2\pi_{11}} \cup U_{\omega_2\pi_{22}} \cup U_{\pi_{11}\pi_{22}} \tag{9}$$

where

$$U_{\omega_1\omega_2} := \left\{ v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \omega_2|_v \neq 0 \right\},$$
$$U_{\omega_1\pi_{11}} := \left\{ v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \pi_{11}|_v \neq 0 \right\},$$
$$U_{\omega_1\pi_{22}} := \left\{ v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \pi_{22}|_v \neq 0 \right\},$$
$$U_{\omega_2\pi_{11}} := \left\{ v \in \pi^{-1}(U) \mid \omega_2|_v \wedge \pi_{11}|_v \neq 0 \right\},$$
$$U_{\omega_2\pi_{22}} := \left\{ v \in \pi^{-1}(U) \mid \omega_2|_v \wedge \pi_{22}|_v \neq 0 \right\},$$
$$U_{\pi_{11}\pi_{22}} := \left\{ v \in \pi^{-1}(U) \mid \pi_{11}|_v \wedge \pi_{22}|_v \neq 0 \right\}.$$

By using this covering, $p^{-1}(U)$ for $p : \Sigma(R) \to R$ is covered as follows:

$$p^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{22}} \cup U_{\omega_2\pi_{11}} \cup U_{\pi_{11}\pi_{22}}. \tag{10}$$

From the covering, we can see that the topological structure of fibers is $T^2 = S^1 \times S^1$.

Rank 2 prolongations of parabolic equations

(R, D); locally parabolic. Then, $\exists$ a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{12}, \pi_{22}\}$ around $x \in R$ s.t.

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} \tag{11}$$

and

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \mod \varpi_0, \\ d\varpi_1 &\equiv \omega_2 \wedge \pi_{12} && \mod \varpi_0, \varpi_1, \varpi_2 \\ d\varpi_2 &\equiv \omega_1 \wedge \pi_{12} + \omega_2 \wedge \pi_{22} && \mod \varpi_0, \varpi_1, \varpi_2 \end{aligned} \tag{12}$$

Theorem 4

Let (R, D) be a locally parabolic equation. Then, the rank 2 prolongation $\Sigma(R)$ has singular points, and it has the structure of pinched torus fibration.

Outline of proof

Similarly to the hyp case, we have the covering of $p : \Sigma(R) \to R$

$$p^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{22}} \cup U_{\pi_{12}\omega_{22}}. \tag{13}$$

Here, $U_{\omega_1\pi_{22}}$ has singular points. By gruing on $p^{-1}(U)$, we have the statement.

Rank 2 prolongations of elliptic equations

(R, D): locally elliptic. Then, $\exists$ a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{11}, \pi_{12}\}$ around $x \in R$ s.t.

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} \tag{14}$$

and

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod} \ \ \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \pi_{11} + \omega_2 \wedge \pi_{12} \quad \text{mod} \ \ \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_1 \wedge \pi_{12} - \omega_2 \wedge \pi_{11} \quad \text{mod} \ \ \varpi_0, \varpi_1, \varpi_2. \end{aligned} \tag{15}$$

Theorem 5

Let (R, D) be a locally elliptic equation. Then, the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D, 2)$, and it is a S^2-bundle over R.

Outline of proof

Similarly to the hyp case, we have the covering of $p : \Sigma(R) \to R$

$$p^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\pi_{11}\pi_{12}}. \tag{16}$$

By gruing on $p^{-1}(U)$, we have the statement.

By using these results, we have:

Corollary 6

Let $R = \{F = 0\}$ be a 2nd order regular PDE and $p : \Sigma(R) \to R$ be a its prolongation. Then,

(i) $w \in R$ is hyperbolic

$\iff$ **$p^{-1}(w)$ is a $2-$dim torus T^2.**

(ii) $w \in R$ is parabolic

$\iff$ **$p^{-1}(w)$ is a pinched $2-$dim torus.**

(iii) $w \in R$ is elliptic

$\iff$ **$p^{-1}(w)$ is a $2-$dim sphere S^2.**

Note that the fiber $p^{-1}(w)$ is defined by the structure equation of D at w as a subset in the fiber $J_w \cong Gr(2,4)$ of the fibration $\pi : J(D,2) \to R$. From this view point, the fiber topology of $p^{-1}(w)$ depends only on the pointwise structure equations.

Geometric singular solution

First, we define the notion of solutions of second order regular PDEs.

Definition 7

(R, D): second order regular PDE. S: 2-dim integral manifold of R. If the restricted projection $\pi_1^2|_R : R \to J^1$ is an immersion on an open dense subset in S, then we call S a geometric solution of (R, D). In particular, if all points of geometric solutions S are immersion points, then we call S regular solutions. On the other hand, geometric solutions S have a nonimmersion point, then we call S singular solutions.

Remark 2 From the definition, images $\pi_1^2(S)$ of geometric solutions S by the projection π_1^2 are Legendrian in $J^1(\mathbb{R}^2, \mathbb{R})$, $(\varpi_0|_{\pi_1^2(S)} = d\varpi_0|_{\pi_1^2(S)} = 0)$.

To consider singular solutions, we consider geometric decomposition of $\Sigma(R)$.

$$\Sigma(R) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \text{ (disjoint union)},$$

where

$$\Sigma_i = \{w \in \Sigma(R) \mid \dim(w \cap \text{fiber}) = i\}$$

$(i = 0, 1, 2)$, fiber means that of $T(R) \supset D \to T(J^1)$.

From the decomposition, we have in general

S: integral manifold of $(\Sigma(R), \hat{D})$.

(1) If $S \subset \Sigma_0$, S is a regular solution.

(2) If S across $\Sigma_1 \cup \Sigma_2$, S is a singular solution.

From this property, we construct explicitly singular solutions of typical equations.

hyperbolic equations

We consider the model equation $\boldsymbol{R} = \{\boldsymbol{s} = 0\}$.

we have the covering of the fibration $\boldsymbol{p} : \Sigma(\boldsymbol{R}) \to \boldsymbol{R}$:

$$\boldsymbol{p}^{-1}(\boldsymbol{U}) = \boldsymbol{U}_{xy} \cup \boldsymbol{U}_{xt} \cup \boldsymbol{U}_{yr} \cup \boldsymbol{U}_{rt},$$

where

$$\boldsymbol{U}_{xy} := \left\{ \boldsymbol{v} \ \in \boldsymbol{\pi}^{-1}(\boldsymbol{U}) \mid \boldsymbol{dx}|_{\boldsymbol{v}} \wedge \boldsymbol{dy}|_{\boldsymbol{v}} \neq 0 \right\},$$
$$\boldsymbol{U}_{xt} := \left\{ \boldsymbol{v} \ \in \boldsymbol{\pi}^{-1}(\boldsymbol{U}) \mid \boldsymbol{dx}|_{\boldsymbol{v}} \wedge \boldsymbol{dt}|_{\boldsymbol{v}} \neq 0 \right\},$$
$$\boldsymbol{U}_{yr} := \left\{ \boldsymbol{v} \ \in \boldsymbol{\pi}^{-1}(\boldsymbol{U}) \mid \boldsymbol{dy}|_{\boldsymbol{v}} \wedge \boldsymbol{dr}|_{\boldsymbol{v}} \neq 0 \right\},$$
$$\boldsymbol{U}_{rt} := \left\{ \boldsymbol{v} \ \in \boldsymbol{\pi}^{-1}(\boldsymbol{U}) \mid \boldsymbol{dr}|_{\boldsymbol{v}} \wedge \boldsymbol{dt}|_{\boldsymbol{v}} \neq 0 \right\}.$$

The geometric decomposition $\Sigma(\boldsymbol{R}) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is given by,

$$\Sigma_0|_{\boldsymbol{p}^{-1}(\boldsymbol{U})} = \boldsymbol{U}_{xy},$$
$$\Sigma_1|_{\boldsymbol{p}^{-1}(\boldsymbol{U})} = (\boldsymbol{U}_{xt} \cup \boldsymbol{U}_{yr}) \backslash \boldsymbol{U}_{xy},$$
$$\Sigma_2|_{\boldsymbol{p}^{-1}(\boldsymbol{U})} = \boldsymbol{U}_{rt} \backslash (\boldsymbol{U}_{xy} \cup \boldsymbol{U}_{xt} \cup \boldsymbol{U}_{yr}).$$

We consider integral submanifolds $\boldsymbol{S}$ across Σ_1 of $\boldsymbol{R} = \{\boldsymbol{s} = 0\}$. Consequently, we obtain the solution of $s = 0$ given by,

$$(x,\ y(t),\ \int qy'dt + z_0(x),\ z_0'(x),$$
$$\int ty'dt,\ z_0''(x),\ t, y',\ z_0'''(x)).$$

where $y(t)$ satisfies the condition $y'(0) = 0$. This is a geometric singular solution.

parabolic equations

We consider the model equation $R = \{r = 0\}$.

we have the covering of the fibration $p : \Sigma(R) \to R$:

$$p^{-1}(U) = U_{xy} \cup U_{xt} \cup U_{st},$$

where

$$U_{xy} := \left\{ v \in \pi^{-1}(U) \mid dx|_v \wedge dy|_v \neq 0 \right\},$$
$$U_{xt} := \left\{ v \in \pi^{-1}(U) \mid dx|_v \wedge dt|_v \neq 0 \right\},$$
$$U_{st} := \left\{ v \in \pi^{-1}(U) \mid ds|_v \wedge dt|_v \neq 0 \right\}.$$

The geometric decomposition $\Sigma(R) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is given by

$$\Sigma_0|_{p^{-1}(U)} = U_{xy},$$
$$\Sigma_1|_{p^{-1}(U)} = U_{xt} \backslash U_{xy},$$
$$\Sigma_2|_{p^{-1}(U)} = U_{st} \backslash (U_{xy} \cup U_{xt}).$$

We consider integral submanifolds S across Σ_1 or Σ_2 of $R = \{r = 0\}$. Consequently, we obtain the solution of $r = 0$ given by

$$(ty' + x_0(s),\ y(s),\ tpy' + z_0(s),$$
$$\int sy'ds,\ tsy' + q_0(s),\ s,\ t,\ x_s,\ y')$$

Here, if $x_0'(0) = y'(0) = z_0'(0) = q_0'(0) = 0$, then we have the singular solution across Σ_2. If $x_0'(0) \neq 0$, $y'(0) = 0$, then we have the singular solution across Σ_1.

elliptic equations

We consider the model equation $R = \{r + t = 0\}$. we have the covering of the fibration $p : \Sigma(R) \to R$:

$$p^{-1}(U) = U_{xy} \cup U_{rs},$$

where

$$U_{xy} := \left\{ v \in \pi^{-1}(U) \mid dx|_v \wedge dy|_v \neq 0 \right\},$$
$$U_{rs} := \left\{ v \in \pi^{-1}(U) \mid dr|_v \wedge ds|_v \neq 0 \right\}.$$

The geometric decomposition $\Sigma(R) = \Sigma_0 \cup \Sigma_2$ is given by

$$\Sigma_0|_{p^{-1}(U)} = U_{xy},$$
$$\Sigma_2|_{p^{-1}(U)} = U_{rs} \backslash U_{xy}.$$

We consider integral submanifolds S across Σ_2 of $R = \{r + t = 0\}$. Consequently, we obtain the solution of $r + t = 0$ given by

$$\begin{aligned}&(ry_s + x_0(s),\ y(r,s),\ \textstyle\int(py_s + qy_r)dr + z_0(s),\\ &\tfrac{r^2}{2}y_s + \textstyle\int sy_r dr + p_0(s),\ rsy_s + \textstyle\int ry_r dr + q_0(s),\\ &\qquad r,\ s,\ y_r,\ y_s),\end{aligned}$$

where $x_0'(0) = z_0'(0) = p_0'(0) = q_0'(0) = 0$ and $y(r,s)$ satisfies the condition $y_r(0,0) = y_s(0,0) = 0$. This is a geometric singular solution across Σ_2.

Tower construction by prolongations.

By succesive rank 2 prolongations, we have the following tower structure of differential systems.

Theorem 8

(i) **If (R, D) is locally hyperbolic, then the k-th rank 2 prolongation $(\Sigma^k(R), \hat{D}^k)$ of (R, D) is also T^2-bundle over $(\Sigma^{k-1}(R), \hat{D}^{k-1})$.**

(ii) **If (R, D) is locally parabolic, then $(\Sigma^k(R) \setminus \{singular\ points\}, \hat{D}^k)$ is also $S^1 \times \mathbb{R}$-bundle over $(\Sigma^{k-1}(R) \setminus \{singular\ points\}, \hat{D}^{k-1})$.**

(iii) **If (R, D) is locally elliptic, then $(\Sigma^k(R), \hat{D}^k)$ of S^2-bundle over $(\Sigma^{k-1}(R), \hat{D}^{k-1})$.**

From this theorem, we have the tower structure by taking prolongations succesively.