Comparison Theorems for ODEs and Their Application to Geometry of Weingarten Hypersurfaces

Takashi Okayasu

Ibaraki University

Differential Geometry and Tanaka Theory
—Differential System and Hypersurface Theory—

Comparison Theorem A

Assume $F(X,\ Y)$ satisfies $F(0,\ 1/y_0) > 0$ and $\partial F/\partial X < 0$ for $X > 0$. Solve the following equations with the initial conditions $x(0) = x_0,\ y(0) = y_0,\ \alpha(0) = \overline{\alpha}(0) = 0$.

$$
\text{(I)} \begin{cases} \dfrac{dx}{ds} = \cos\alpha, \\ \dfrac{dy}{ds} = \sin\alpha, \\ \dfrac{d\alpha}{ds} = F\left(\dfrac{\sin\alpha}{x},\ \dfrac{\cos\alpha}{y}\right). \end{cases}
$$

$$
\text{(II)} \begin{cases} \dfrac{dx}{ds} = \cos\overline{\alpha}, \\ \dfrac{dy}{ds} = \sin\overline{\alpha}, \\ \dfrac{d\overline{\alpha}}{ds} = F\left(0,\ \dfrac{\cos\overline{\alpha}}{y}\right). \end{cases}
$$

$$\Longrightarrow \quad \alpha(y) < \overline{\alpha}(y) \quad \text{for all } y \text{ satisfying } \ 0 < \alpha(y) < \frac{\pi}{2}$$

Comparison Theorem B

Assume $F(0,\ 1/y_0) > 0$ and $\partial F/\partial X,\ \partial F/\partial Y < 0$ for $X,\ Y > 0$. Solve the following equations with the initial conditions $x(0) = x_0,\ y(0) = y_0,\ \alpha(0) = \overline{\alpha}(0) = \alpha_0 = 0$.

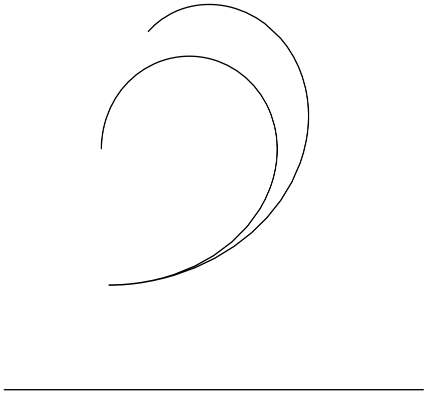
$$\text{(I)} \begin{cases} \frac{dx}{ds} = \cos\alpha, \\ \frac{dy}{ds} = \sin\alpha, \\ \frac{d\alpha}{ds} = F(\frac{\sin\alpha}{x},\ \frac{\cos\alpha}{y}). \end{cases}$$

$$\text{(II)} \begin{cases} \frac{dx}{ds} = \cos\overline{\alpha}, \\ \frac{dy}{ds} = \sin\overline{\alpha}, \\ \frac{d\overline{\alpha}}{ds} = F(\frac{\sin\overline{\alpha}}{x_0},\ \frac{\cos\alpha_0}{y}). \end{cases}$$

$\Longrightarrow \quad \alpha(y) > \overline{\alpha}(y)$ for all y satisfying $0 < \alpha(y) < \frac{\pi}{2}$

Example

$$F(X, Y) = 1 - X - Y$$



Our Aim

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- constant mean curvature hypersurfaces $\lambda_1 + \cdots + \lambda_n =$const
- constant scalar curvature hypersurfaces $\sum_{i \neq j} \lambda_i \lambda_j =$const
- hypersurfaces whose second fundamental form h have constant length $|h|^2 = \sum_i \lambda_i^2 =$const

Delauney surfaces

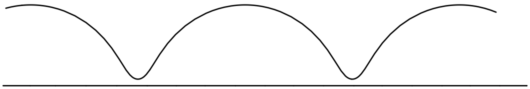


Figure: unduloid

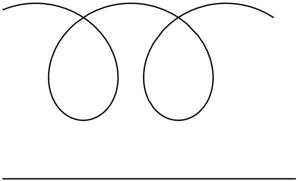
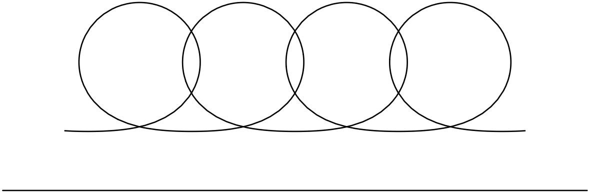


Figure: nodoid

A rotational surface with $|h| = \mathrm{const}$



Noncompact Complete Hypersurfaces with Constant Scalar Curvature

Known Examples

- flat generalized cylinders
- cylinders $S^p \times \mathbf{R}^{n-p}$
- 1-parameter family of rotational hypersurfaces (Leite, 1990)
- a complete hypersurface with constant negative scalar curvature in E^4 (O, 1989)
- a complete hypersurface with 0 scalar curvature in E^4 (Palmas,2000)
- a complete hypersurface with 0 scalar curvature in E^{2n} (Sato, 2000)

Result

We can construct a new family of complete hypersurfaces with constant positive scalar curvature. They are diffeomorphic to $\mathbf{R} \times S^p \times S^q$.

Result & Method

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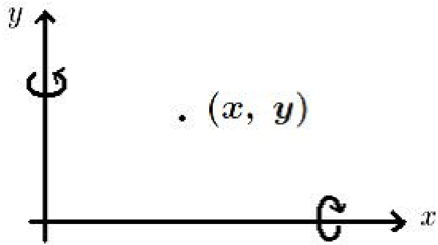
Method

- Method of equivariant geometry
  We use a subgroup
  $O(p+1) \times O(q+1) \subset O(p+q+2) = O(n+1)$ to construct Generalized rotational hypersurfaces.
  PDE $\Rightarrow$ ODE
- This method was used by W.Y. Hsiang et al. to construct many CMC hypersurfaces in the 80's.

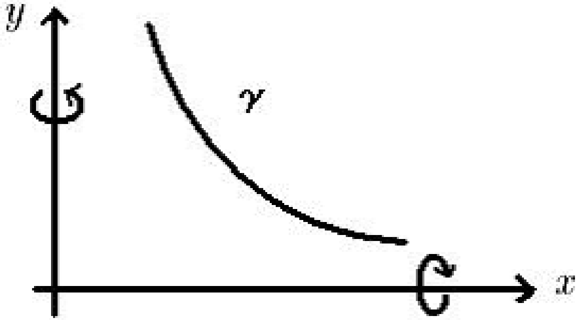
$O(p+1) \times O(q+1)$-invariant hypersurfaces

- $O(p+1) \times O(q+1) \curvearrowright \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$
- The orbit space= the first quadrant $\mathbf{R}^2_+$
- The orbit through $(x,\ y) = S^p(x) \times S^q(y)$



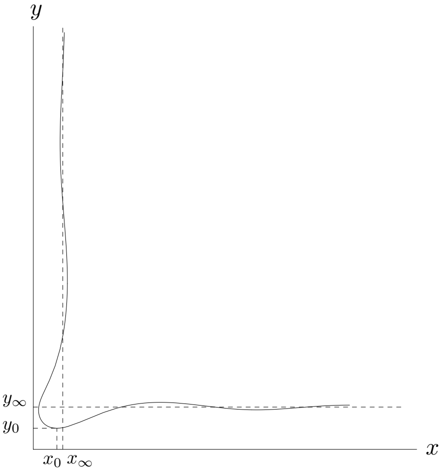
Generalized Rotational Hypersurfaces

γ : a curve in the first quadrant $\mathbf{R}^2_+$
M_γ : $O(p+1) \times O(q+1)$-invariant hypersurface generated by γ



Main Theorem 1

There exists a new family of complete hypersurfaces $M^n \subset E^{n+1}$ $(n \geq 5)$ with constant positive scalar curvature.



Scalar Curvature Equation

principal curvatures

$x'y'' - y'x''$, $\frac{y'}{x}$, $-\frac{x'}{y}$: multiplicities 1, p, q
(the curve is parametrized by the arc length)

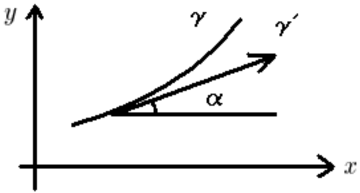
The scalar curvature of M_γ

$$\begin{aligned} S = & \sum_{i \neq j} \lambda_i \lambda_j \\ = & 2(x'y'' - y'x'') \left(p\frac{y'}{x} - q\frac{x'}{y} \right) + p(p-1)\left(\frac{y'}{x}\right)^2 \\ & + q(q-1)\left(\frac{x'}{y}\right)^2 - 2pq\frac{y'}{x}\frac{x'}{y} \end{aligned}$$

Constant Scalar Curvature Equation

$S =$ constant

$$
\text{(I)}\begin{cases}
\dfrac{dx}{ds} = \cos\alpha, \\
\dfrac{dy}{ds} = \sin\alpha, \\
\dfrac{d\alpha}{ds} = \dfrac{p(p-1)\left(\frac{\sin\alpha}{x}\right)^2 - 2pq\frac{\sin\alpha}{x}\frac{\cos\alpha}{y} + q(q-1)\left(\frac{\cos\alpha}{y}\right)^2 - S}{2\left(q\frac{\cos\alpha}{y} - p\frac{\sin\alpha}{x}\right)}.
\end{cases}
$$

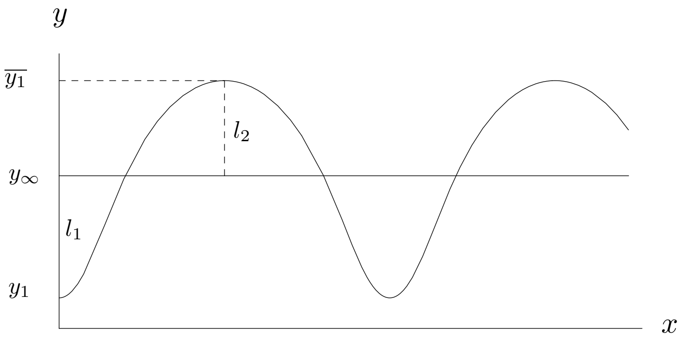


Comparison Equation

$$(\mathrm{I})\begin{cases} \dfrac{dx}{ds} = \cos\alpha, \\ \dfrac{dy}{ds} = \sin\alpha, \\ \dfrac{d\alpha}{ds} = \dfrac{p(p-1)\left(\frac{\sin\alpha}{x}\right)^2 - 2pq\frac{\sin\alpha}{x}\frac{\cos\alpha}{y} + q(q-1)\left(\frac{\cos\alpha}{y}\right)^2 - S}{2\left(q\frac{\cos\alpha}{y} - p\frac{\sin\alpha}{x}\right)}. \end{cases}$$

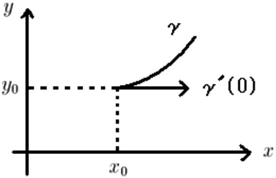
$$(\mathrm{II})\begin{cases} \dfrac{dx}{ds} = \cos\alpha, \\ \dfrac{dy}{ds} = \sin\alpha, \\ \dfrac{d\alpha}{ds} = \dfrac{q(q-1)\left(\frac{\cos\alpha}{y}\right)^2 - S}{2q\frac{\cos\alpha}{y}}. \end{cases}$$

The solution of (II)

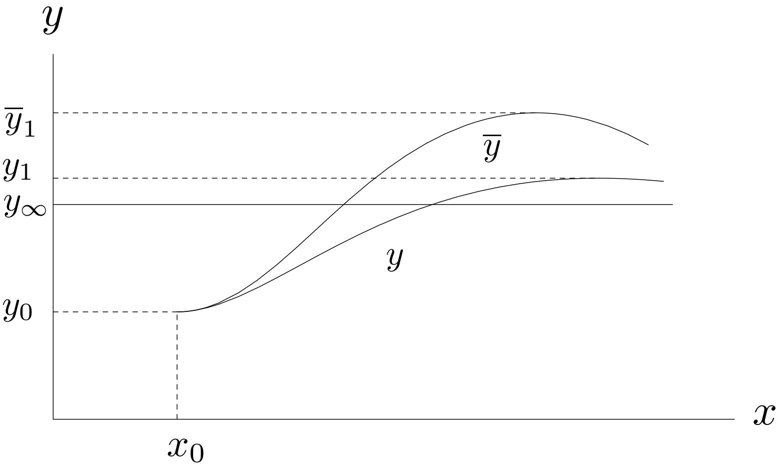


- $y_0 < y_\infty = \sqrt{q(q-1)/S}$
- Solve (I), (II) with the same initial conditions $x(0) = x_0,\ y(0) = y_0,\ \alpha(0) = 0.$

$\Rightarrow \alpha(y) < \overline{\alpha}(y)$ where $\overline{\alpha}$ is the solution of (II)

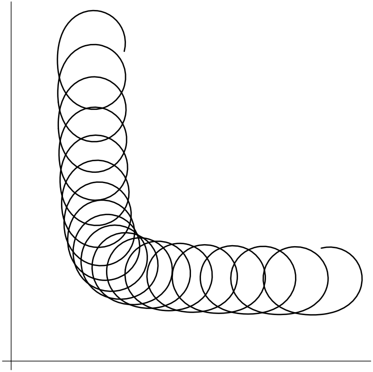


Comparison of Solution Curves



Main Theorem 2

There exists a new family of complete hypersurfaces $M^n \subset E^{n+1}$ $(n \geq 5)$ with $|h| =$const.



Comparison Theorem for $|h| = h_0$(const)

Solve the following equations with the initial conditions
$x(0) = x_0,\ y(0) = y_0,\ \alpha(0) = \overline{\alpha}(0) = \alpha_0.$

$$
\text{(I)} \begin{cases} \frac{dx}{ds} = \cos\alpha, \\ \frac{dy}{ds} = \sin\alpha, \\ \frac{d\alpha}{ds} = \sqrt{h_0^2 - p\left(\frac{\sin\alpha}{x}\right)^2 - q\left(\frac{\cos\alpha}{y}\right)^2}. \end{cases}
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\text{(II)} \begin{cases} \frac{dx}{ds} = \cos\alpha, \\ \frac{dy}{ds} = \sin\alpha, \\ \frac{d\alpha}{ds} = \sqrt{h_0^2 - p\left(\frac{\sin\alpha}{x_0}\right)^2 - q\left(\frac{\cos\alpha_0}{y}\right)^2}. \end{cases}
$$

$$
\Longrightarrow \quad \alpha(y) > \overline{\alpha}(y), \quad \text{when } 0 < \alpha < \frac{\pi}{2}
$$

Theorem (Ye, 1991) Let (M^{n+1}, g) be a Riemannian manifold. Suppose that p_0 is a nodegenerate critical point of the scalar curvature of M.Then there exists $r_0 > 0$, such that for all $\rho \in (0, r_0)$, the geodesic sphere $S_\rho(p_0)$ may be perturbed to a constant mean curvature hypersurface S_ρ with $H = 1/\rho$.

Theorem *Let (M^{n+1}, g) be a Riemannian manifold. Suppose that p_0 is a nondegenerate critical point of the scalar curvature of M. Then there exists $r_0 > 0$, such that for all $\rho \in (0, r_0)$, the geodesic sphere $S_\rho(p_0)$ may be perturbed to a closed hypersurface S_ρ whose second fundamental form are of constant length $\sqrt{n}/\rho$.*