Moment map of the Spin action and the Cartan-Münzner Polynomials

Reiko MIYAOKA (Tohoku University)

Differential Geometry and Tanaka Theory

$\sim$ Differential System and Hypersurface Theory $\sim$

(RIMS/Kyoto, 26 Jan. 2011)

1 Introduction

A family of isoparametric hypersurfaces M_t (hypersurfaces with constant principal curvatures) in $S^n \subset \mathbb{R}^{n+1}$

$$\Updownarrow \quad M_t = F^{-1}(t) \cap S^n$$

A Cartan-Münzner Polynomial $F(x)$ on $\mathbb{R}^{n+1}$

- Isop. h's. are algebraic, but <u>not necessarily homogeneous.</u>

Goal. Express $F(x)$ by using a moment map of a certain group action.

Most interesting case : deg $F(x) = 4$

$\exists$ isop. h's. of OT-FKM type obtained from the representations of Clifford algebras.

- This type includes **infinitely many homog. and non-homog. isop. h's.**, on which $\exists$ a Spin action.

Conclusion. We give an expression of $F(x)$ in terms of the moment map of the Spin action extended to $T\mathbb{R}^{n+1}$.

2 Preliminaries

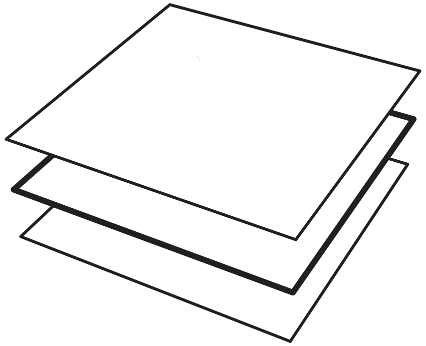
Definition. M^{n-1} : a hypersurface in $\overline{M} = \mathbb{R}^n, H^n, S^n$
is **an isoparametric hypersurface**

$\Leftrightarrow M$ has constant principal curvatures.

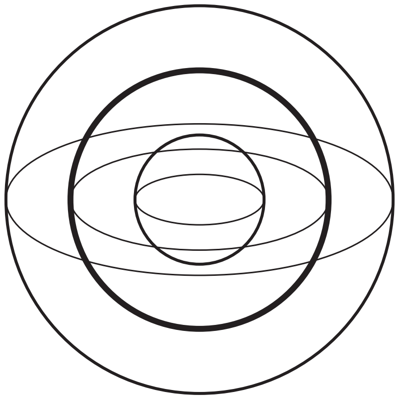
The surface case $\Rightarrow$ trivial

Consider the hypersurface case $(n \geq 4)$.

Examples

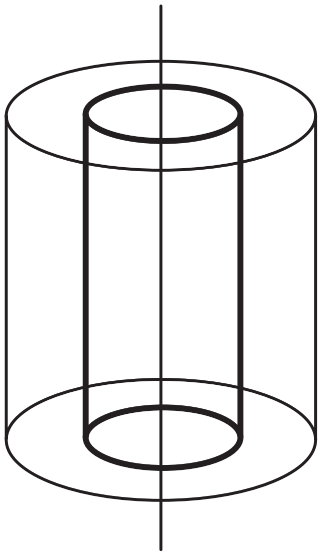


$E^{n-1} \subset E^n$
$H^{n-1} \subset H^n$



$S^{n-1} \subset E^n, H^n$

$S^{k-1} \times E^{n-k}$
$S^{k-1} \times H^{n-k}$



$\overline{M}$	M^{n-1}		
$\mathbb{R}^n$	$\mathbb{R}^{n-1}$ or S^{n-1}	$\mathbb{R}^k \times S^{n-k-1}$	–
H^n	H_{eq} or S^{n-1}	$H^k_{eq} \times S^{n-k-1}$	–
S^n	S^{n-1}	$S^k \times S^{n-k-1}$	more

H_{eq} : an equidistant h's, including a horosphere.

{homogeneous h'surfaces} $\subset$ {isoparametric h'surfaces}

The equality holds for $\mathbb{R}^n$ and H^n. (É. Cartan, '37).

• There exist more homogeneous and non-homogeneous examples in S^n, <u>most of the cases have been classified.</u>

Fact 1. (Münzner, '81) *For isop. h's. M_t in S^n :*

(a) $g = \sharp\{$*distinct principal curvatures*$\} \in \{1, 2, 3, 4, 6\}$.

(b) *For the principal curvatures $\lambda_1 > \lambda_2 > \cdots > \lambda_g$, the multiplicities $m_1, m_2, \ldots, m_g$ satisfy $m_i = m_{i+2}$.*

(c) *There exists a* **Cartan-Münzner polynomial** $F : \mathbb{R}^{n+1} \to \mathbb{R}$, *homogeneous and of degree g, such that*

$$\begin{aligned} &(i) \quad \|DF(x)\|^2 = g^2 \|x\|^{2g-2} \\ &(ii) \quad \triangle F(x) = \frac{m_2 - m_1}{2} g^2 \|x\|^{g-2}, \end{aligned} \tag{1}$$

and $M_t = F^{-1}(t) \cap S^n$, $-1 < t < 1$.

Classification of isoparametric h's. in S^n:

g	1	2	3	4^*	6
M	S^{n-1}	$S^k \times S^{n-k-1}$	$C_{\mathbb{F}}$ (hom.)	homog. or OT-FKM	N^6, M^{12} (hom.)

$[g \leq 3]$: Cartan. $[g = 4]$: Cecil-Chi-Jensen, Immervoll except for 4 cases. $[g = 6]$: Dorfmeister-Neher, Miyaoka.

- Non-homogeneous case occurs only when $g = 4$.

Known isoparametric hypersurfaces in S^n with $g = 4$:

<table>
<tr><td rowspan="3">OT-FKM type</td><td colspan="2">non-homogeneous $(m_1, m_2) = (3, 4k), (7, 8k)$, etc.</td></tr>
<tr><td rowspan="4">homogeneous: isotropy orbits of G/K</td><td>G/K : non-Hermitian $(4, 4k-1)$</td></tr>
<tr><td>*Hermitian $(1, k), (2, 2k-1), (9, 6)$</td></tr>
<tr><td rowspan="2">non OT-FKM</td><td>*Hermitian $(4, 5)$</td></tr>
<tr><td>non-Hermitian $(2, 2)$</td></tr>
</table>

They are all classfied except for $(m_1, m_2) = (4, 5), (6, 9), (7, 8)$

(Cecil-Chi-Jensen, Immervoll, and Chi for $(3, 4)$).

Fact 2. (Hsiang-Lawson, '69) *Homogeneous h's. in* S^n *are given by isotropy orbits of rank two symmetric spaces.*

Fact 3. (Fujii, F.-Tamaru, '10) *Homogeneous h's. associated with Hermitian symmetric spaces* G/K *have Cartan-Münzner polynomials* $F(x)$ *expressed by the square norm of the moment map of the isotropy action.*

Problem. What about the case of non-homogeneous, and homogeneous ones associated with non-Hermitian symmetric space?

$\Rightarrow$ **Need a new idea.**

3 Clifford systems on $\mathbb{R}^{2l}$, OT-FKM type

$O(n)$: the orthogonal group, $\mathfrak{o}(n)$: its Lie algebra.

Definition. (1) $\mathcal{C}_{m-1} = \{E_1, \ldots, E_{m-1}\}$, $E_j \in \mathfrak{o}(l)$:

a representation of a Clifford algebra

$$\Leftrightarrow E_i E_j + E_j E_i = -2\delta_{ij}\mathrm{id}, \quad 1 \leq i, j \leq m-1.$$

(2) $P_0, \ldots P_m \in O(2l)$, symmetric : **a Clifford system**

$$\Leftrightarrow P_i P_j + P_j P_i = 2\delta_{ij}\mathrm{id}, \quad 0 \leq i, j \leq m.$$

Lemma 3.1 *There exists a one-to-one correspondence between $\mathcal{C}_{m-1}$ and the Clifford system.*

Proof: From $\mathcal{C}_{m-1}$, putting $(u,v) \in \mathbb{R}^l \oplus \mathbb{R}^l$, we obtain

$$P_0(u,v) = (u,-v),\ P_1(u,v) = (v,u),\ P_{1+i}(u,v) = (E_i v, -E_i u),$$

which satisfy (2).

From a Clifford system $P_0, \dots P_m$, decompose $\mathbb{R}^{2l} = E_+(P_0) \oplus E_-(P_0)$ where $E_\pm(P_0)$ is the ± 1 eigenspace of P_0. Then $E_i = P_1 P_{1+i} \in \mathfrak{o}(l)$, $i = 1, \dots m-1$, satisfy (1).

$\square$

Remark 3.2 : (1) The possible pairs (m, l) :

m	1	2	3	4	5	6	7	8	$\cdots$	$m+8$	$\cdots$
$l=\delta(m)$	1	2	4	4	8	8	8	8	$\cdots$	$16\delta(m)$	$\cdots$

(2) W.r.t. the inner product of linear operators on $\mathbb{R}^{2l}$

$$\langle P, Q\rangle = \frac{1}{2l}\mathrm{Tr}(P^t Q). \tag{2}$$

$P_0, \ldots, P_m$ give an orthonormal basis of the linear space V of symmetric orthogonal operators, which they span.

Fact 4. (Ferus-Karcher-Münzner '81)

When a Clifford system $P_0, \dots, P_m$ is given,

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2 \tag{3}$$

is a Cartan-Münzner polynomial of degree four. If $l - m - 1 > 0$, $F|_{S^{2l-1}}$ defines isoparametric hypersurfaces in S^{2l-1} with $g = 4$ and $m_1 = m$, $m_2 = l - m - 1$.

Q. Can we express $F(x)$ using a moment map of a certain group action?

When a Clifford system $P_0, \ldots, P_m$ is given, P_iP_j, $0 \leq i < j \leq m$, are skew, and generate a Lie subalgebra of $\mathfrak{o}(2l)$ isomorphic to $\mathfrak{o}(m+1)$.

Fact 5. [FKM, '81] *$\underline{Spin(m+1)\ acts}$ on $\mathbb{R}^{2l}$, and preserves $F(x)$, i.e., is constant on each level set.*

Remark 3.3 : If there exists a moment map μ of this action, $\|\mu\|$ is constant along each orbit, and so seems to relate with $F(x)$. In order to make this action Hamiltonian, we have to extend it to an action on the symplectic manifold $T\mathbb{R}^{2l}$.

4 Review of symplectic geometry

Definition.

(1) (M^{2n}, ω) is **a symplectic manifold**

$\Leftrightarrow$ ω is a non-degenerate closed 2-form on M.

(2) **The Hamiltonian vector field** H_f **of** $f \in \mathbb{C}^\infty(M)$

$\Leftrightarrow df = \omega(H_f, \ )$.

Put $\mathrm{Ham}(M) = \{H_f \mid f \in \mathbb{C}^\infty(M)\}$.

K : a compact Lie group acting on M.

Definition. (1) **a fundamental vector field** on M

$$\Leftrightarrow X_\zeta = \frac{d}{dt}\Big|_{t=0} (\exp t\zeta)x, \quad \zeta \in \mathfrak{k}$$

(2) $K \curvearrowright M$ is **a symplectic action**

$$\Leftrightarrow k^*\omega = \omega, \forall k \in K.$$

(3) $K \curvearrowright M$ is **a Hamiltonian action**

$$\Leftrightarrow X_\zeta \in \mathrm{Ham}(M), \forall \zeta \in \mathfrak{k}.$$

i.e., $\exists \mu_\zeta \in \mathbb{C}^\infty(M)$ s.t. $d\mu_\zeta = \omega(X_\zeta, \ )$.

(4) With respect to the coadjoint action of K on $\mathfrak{k}^*$,

$\mu : M \to \mathfrak{k}^*$ is **a moment map**

$$\Leftrightarrow \begin{array}{l} \text{(i) } \mu \text{ is } K \text{ equivariant} \\ \text{(ii) } d\mu(\zeta) = \omega(X_\zeta, \ ) \end{array}$$

- $K \curvearrowright M$ is Hamiltonian

$$\Leftrightarrow \exists \mu : M \to \mathfrak{k}^*, \text{ the moment map}$$

Example. (1) $(\mathbb{C}^n, J, \omega)$ with $\omega(X, \ ) = -\langle JX, \ \rangle$

$K \curvearrowright \mathbb{C}^n$: Hamiltonian $\Rightarrow d\mu_\zeta(Y) = \omega(X_\zeta, Y) = -\langle JX_\zeta, Y\rangle \Rightarrow \nabla\mu_\zeta = -JX_\zeta$, or $X_\zeta = J\nabla\mu_\zeta$.

(2) G/K : a Hermitian symmetric space,

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition,

$\exists$ a center $\mathfrak{c}$ of $\mathfrak{k} \Rightarrow \exists$ a Kähler structure J on $\mathfrak{p}$ given by

$$Jx = \mathrm{ad}\mathfrak{z}(x) = -\mathrm{ad}x(\mathfrak{z}), \quad \mathfrak{z} \in \mathfrak{c},\ x \in \mathfrak{p}.$$

$\Rightarrow$ the isotropy action $K \curvearrowright \mathfrak{p}$ is a Hamiltonian action with **the moment map** $\mu^F(x) = \frac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}$ **(Ohnita)**.

Fact 3. (Fujii, F-Tamaru, '10) *When an isoparametric hypersurface is given by an isotropy orbit of a Hermitian symmetric space* G/K*, the Cartan-Münzner polynomial is given by*

$$F(x) = a\|\mu_0(x)\|^2 - b\|\mu_1(x)\|^2.$$

where $\mu^F = \mu_0 + \mu_1 : \mathfrak{p} \to \mathfrak{k}^*$ *is the moment map decomposed into the components of* $\mathfrak{k}^* = \mathfrak{c}^* \oplus \mathfrak{k}_1^*$*, and* a, b *are constants depending on* m_1, m_2*.*

5 $Spin(m+1)$ action on $T\mathbb{R}^{2l}$

A complex structure $\tilde{J}$ on $T\mathbb{R}^{2l}$:

$$\tilde{J}(U,V) = (-V,U), \quad (U,V) \in T_{(x,Y)}(T\mathbb{R}^{2l}) \cong \mathbb{R}^{2l} \oplus \mathbb{R}^{2l}.$$

$\Rightarrow T\mathbb{R}^{2l}$: a symplectic manifold with a symplectic form

$$\omega(\tilde{Z},\tilde{W}) = -\langle \tilde{J}\tilde{Z},\tilde{W}\rangle, \quad \tilde{Z},\tilde{W} \in T_{(x,Y)}(T\mathbb{R}^{2l}),$$

$\tilde{J}$: parallel $\Rightarrow \omega$ is a non-degenerate closed 2-form.

$P_0, \ldots, , P_m$: Clifford system on $\mathbb{R}^{2l}$

$\Rightarrow \zeta_{ij} = P_i P_j \in \mathfrak{o}(2l)$, $0 \le i < j \le m$, generate $\mathfrak{o}(m+1)$, acting on $\mathbb{R}^{2l}$. The $Spin(m+1)$ action on $\mathbb{R}^{2l}$ is given by $(\exp tP_iP_j)x$ for $x \in \mathbb{R}^{2l}$, and is extended to $T\mathbb{R}^{2l}$ by

$$s \cdot (x, Y) = (sx, sY), \quad s \in Spin(m+1),\, (x, Y) \in T_x\mathbb{R}^{2l}.$$

The fundamental vector field $\tilde{X}_\zeta$ is given by

$$\tilde{X}_\zeta = (x, Y; \zeta x, \zeta Y) \in T_{(x,Y)}(T\mathbb{R}^{2l}), \quad \zeta \in \mathfrak{o}(m+1)$$

which we abbreviate to $\tilde{X}_\zeta = (\zeta x, \zeta Y)$.

Proposition 5.1 *$Spin(m+1) \curvearrowright T\mathbb{R}^{2l}$ is symplectic.*

Proof: Since

$$\tilde{J}(U,V) = (-V,U), \quad (U,V) \in T_{(x,Y)}(T\mathbb{R}^{2l}) \cong \mathbb{R}^{2l} \oplus \mathbb{R}^{2l},$$

$\tilde{J}$ is the right action of $\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{u}(1) \cong \mathfrak{o}(2)$, i.e.,

$\tilde{J}(U,V) = (U,V)\eta$.

$\Rightarrow$ commutes with $\zeta = \zeta_{ij} = P_i P_j \in \mathfrak{k}$, $(0 \leq i < j \leq m)$.

$\square$

Theorem 5.2 $Spin(m+1) \curvearrowright T\mathbb{R}^{2l}$ is Hamiltonian.

Proof: The moment map $\mu : T\mathbb{R}^{2l} \to \mathfrak{o}^*(m+1)$ is given by, for $\zeta_{ij} = P_i P_j, 0 \leq i < j \leq m$,

$$\begin{aligned}
\mu((x,Y))(\zeta_{ij}) &= \tfrac{1}{2}\langle \zeta_{ij}(x,Y), \tilde{J}(x,Y)\rangle \\
&= \tfrac{1}{2}\langle (\zeta_{ij}x, \zeta_{ij}Y), (-Y,x)\rangle \qquad (4) \\
&= \langle \zeta_{ij}Y, x\rangle = -\langle \zeta_{ij}x, Y\rangle,
\end{aligned}$$

In fact, this is equivariant w.r.t. the coadjoint action $Spin(m+1) \curvearrowright \mathfrak{o}^*(m+1)$, since for $s \in Spin(m+1)$,

$$\begin{aligned}
\mu(s \cdot (x,Y))(\zeta_{ij}) &= \langle \zeta_{ij}(sY), sx \rangle = \langle s(s^{-1}\zeta_{ij}s)Y, sx \rangle \\
&= \langle (s^{-1}\zeta_{ij}s)Y, x \rangle = \langle \Big((Ad_s)^{-1}\zeta_{ij} \Big) Y, x \rangle \\
&= \mu(x,y) \circ (Ad_s)^{-1}\zeta_{ij} \\
&= (Ad^*)_s(\mu(x,Y))(\zeta_{ij}).
\end{aligned}$$

Then for $\tilde{Z} \in T_{(x,Y)}T\mathbb{R}^{2l}$, using $\tilde{J}\zeta_{ij} = \zeta_{ij}\tilde{J}$, we obtain

$$d\mu(\zeta_{ij})(\tilde{Z}) = \langle \tilde{X}_{ij}, \tilde{J}\tilde{Z} \rangle = -\langle \tilde{J}\tilde{X}_{ij}, \tilde{Z} \rangle = \omega(\tilde{X}_{ij}, \tilde{Z}).$$

Thus μ is the moment map. $\square$

We may regard ζ_{ij} as an orthonormal frame of $\mathfrak{o}(m+1)$.

Corollary 5.3 The moment map of the $Spin(m+1)$ action on $T\mathbb{R}^{2l}$ is given by

$$\mu(x,Y) = -\sum_{0\leq i<j\leq m} \langle \zeta_{ij} x, Y\rangle \zeta_{ij} \in \mathfrak{o}(m+1) \cong \mathfrak{o}^*(m+1).$$

From this follows $\|\mu(x,Y)\|^2 = \sum_{0\leq i<j\leq m} \langle P_i P_j x, Y\rangle^2$.

Since the $U(1) \curvearrowright T\mathbb{R}^{2l}$ induced by η is commutative, this action is also Hamiltonian.

6 Main Theorem

Theorem 6.1 *$P_0, \ldots, P_m$ on $\mathbb{R}^{2l}$: a Clifford system,*
$Y : \mathbb{R}^{2l} \to T\mathbb{R}^{2l}$: (not necessarily continuous) vector field;

$$Y_x = \begin{cases} P_0 x, & \text{if } \langle P_0 x, x\rangle = 0 \\ \dfrac{\langle P_1, x, x\rangle P_0 x - \langle P_0 x, x\rangle P_1 x}{\sqrt{\langle P_1 x, x\rangle^2 + \langle P_0 x, x\rangle^2}}, & \text{if } \langle P_0 x, x\rangle \neq 0. \end{cases}$$

$\Rightarrow$

$$F(x) = \|\mu_0(x, Y_x)\|^2 - 2\|\mu(x, Y_x)\|^2,$$

where $\mu_0 + \mu$ is the moment map of $U(1) \times Spin(m+1) \curvearrowright T\mathbb{R}^{2l}$.

Remark 6.2 : We use the moment map on $T\mathbb{R}^{2l}$, but the RHS is determined by $x \in \mathbb{R}^{2l}$.

Remark 6.3 : P_0, P_1 can be replaced by any two orthogonal unit elements of V. This corresponds to that there is no standard choice of a principal vector for λ_1 if $m_1 > 1$.

Remark 6.4 : $C = \{(x, Y_x) \in T\mathbb{R}^{2l}\}$ is a $2l$ dimensional cone outside x such that $\langle P_0 x, x\rangle = 0$. However, C is not a Lagrangian cone of $T\mathbb{R}^{2l}$.

7 Homogeneous case

$F(x)$ for the OT-FKM type has been expressed by μ:

<table>
<tr><td rowspan="3">OT-FKM type</td><td>non-homogeneous</td><td>$(m_1, m_2) = (3, 4k), (7, 8k)$, etc.</td></tr>
<tr><td rowspan="4">homogeneous: isotropy orbits of G/K</td><td>G/K : non-Hermitian $(4, 4k-1)$</td></tr>
<tr><td>*Hermitian $(1, k), (2, 2k-1), (9, 6)$</td></tr>
<tr><td rowspan="2">non OT-FKM</td><td>*Hermitian $(4, 5)$</td></tr>
<tr><td>non-Hermitian $(2, 2)$</td></tr>
</table>

Since there are two homogeneous h's. not of OT-FKM type, we review homogeneous cases in general :

G/K : a rank two symmetric space

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the Cartan decomposition

Extend the isotropy action $K \curvearrowright \mathfrak{p}$ to $T\mathfrak{p}$ in a natural way:

$k \cdot (x, Y) = (\mathrm{Ad}k(x), \mathrm{Ad}k(Y)), \quad (x, Y) \in T\mathfrak{p},\ k \in K.$

• On $T\mathfrak{p}$, we define the symplectic struture as before, since $\mathfrak{p} \cong \mathbb{R}^{2l}$.

Proposition 7.1 *G/K : a rank two symmetric space,*
$\Rightarrow U(1) \times K \curvearrowright T\mathfrak{p}$ is a Hamiltonian action with the moment map $\mu_0 + \mu : T\mathfrak{p} \to \mathfrak{u}(1)^* \oplus \mathfrak{k}^*$;

$$\begin{aligned} \mu_0(x, Y) &= \frac{1}{2}(\|x\|^2 + \|Y\|^2)\eta, \\ \mu(x, Y) &= -\mathrm{ad}x(Y), \quad (x, Y) \in T\mathfrak{p}. \end{aligned}$$

Corollary 7.2 *If G/K is a Hermitian symmetric space, for $\mathfrak{z} \in \mathfrak{c} \subset \mathfrak{k}$ s.t. $J = \mathrm{ad}\mathfrak{z}$*

$$\Rightarrow \mu(x, \tfrac{1}{2}Jx) = \mu^F(x) = \tfrac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}$$

Remark 7.3 : The proposition holds not only for $g = 4$, but also for all the homogeneous hypersurfaces.

Remark 7.4 : In the OT-FKM case (which cotains some homogeneous case), we gave an expression of $F(x)$ via the moment map of the $Spin(m+1)$ action which is **smaller than** K.

8 Case not of OT-FKM type

Fact 6. **(FKM)** *The isotropy orbits of* $SO(5) \times SO(5)/SO(5)$, $(m_1, m_2) = (2, 2)$, *and of the Hermitian symmetric space* $SO(10)/U(5)$,$(m_1, m_2) = (4, 5)$, *are not of OT-FKM type, and these are all the known examples.*

In this case, we express $F(x)$ by the moment map of the isotropy action of K extended to $T\mathfrak{p}$.

For $G/K = SO(5) \times SO(5)/SO(5)$, or $SO(10)/U(5)$,

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the Cartan decomposition,

$\mathfrak{a} \subset \mathfrak{p}$: a maximal abelian subspace,

Put $G_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}(5) \subset \mathfrak{u}(5)$, $1 \leq i < j \leq 5$, where, E_{ij} is the matrix with (i, j) component equal to one and all other components equal to 0.

Theorem 8.1 *When* $(m_1, m_2) = (2,2),\ (4,5)$ *which are not of OT-FKM, using* $\tau = G_{25} + G_{45} \in \mathfrak{k}$*, put* $Y_H = [H, \tau] \in \mathfrak{p}$ *for* $H \in \mathfrak{a}$*, and extend it to a vector field* Y_x *on* $\mathfrak{p}$ *by the action of* K*. Restricting* $\mu_0 + \mu$ *to the cone* $C = \{(x, Y_x) = \mathrm{Ad}k(H, Y_H)\} \subset T\mathfrak{p}$*, we can express*

$$F(x) = p\|\mu_0(x, Y_x)\|^2 - q\|\mu(x, Y_x)\|^2,$$

where $(p, q) = (3, 4)$ *for* $(m_1, m_2) = (2, 2)$*, and* $(p, q) = (\frac{3}{4}, 1)$ *for* $(m_1, m_2) = (4, 5)$*.*

Proof: By using the root systems, we can take

$$H = \sqrt{-1}H(\xi_1, \xi_2) = \sqrt{-1}(\xi_1 G_{12} + \xi_2 G_{34}) \in \mathfrak{a} \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Then G_{25} and G_{45} satisfy

$$(\mathrm{ad}H)^2 G_{25} = -\xi_1^2 G_{25}, \quad (\mathrm{ad}H)^2 G_{45} = -\xi_2^2 G_{45}, \quad H \in \mathfrak{a}.$$

$$\Downarrow$$

$$\mu(H, Y_H) = -(\mathrm{ad}H)^2 \tau = \xi_1^2 G_{25} + \xi_2^2 G_{45}$$

$$\Downarrow$$

$$\|\mu(H, Y_H)\|^2 = \xi_1^4 + \xi_2^4.$$

We have

$$\|H\|^2 = \xi_1^2 + \xi_2^2,$$

and from $Y_H = \xi_1 G_{15} + \xi_2 G_{35}$,

$$\|Y_H\|^2 = \xi_1^2 + \xi_2^2.$$

Thus from (7.1),

$$\|\mu_0(x, Y_x)\|^2 = \|\mu_0(H, Y_H)\|^2 = (\xi_1^2 + \xi_2^2)^2.$$

On the other hand, $F(x)$ is given by Ozeki-Takeuchi,

$$F(x) = \frac{3}{4}(\mathrm{Tr}(x^2))^2 - 2\mathrm{Tr}x^4, \quad x \in \mathfrak{p}.$$

For $x = H \in \mathfrak{a}$, and $\mathrm{id}_{ij} = E_{ii} + E_{jj}$,

$$H^2 = \xi_1^2 \mathrm{id}_{12} + \xi_2^2 \mathrm{id}_{34}, \quad H^4 = \xi^4 \mathrm{id}_{12} + \xi_2^4 \mathrm{id}_{34}.$$

hold, and we obtain

$$F(H) = 3(\xi_1^2 + \xi_2^2)^2 - 4(\xi_1^4 + \xi_2^4) = 3\|\mu_0(x, Y_x)\|^2 - 4\|\mu(x, Y_x)\|^2, \tag{5}$$

which is constant on the orbit through H. $\square$

9 Summary

Finally, the Cartan-Münzner polynomilas with $g = 4$ *are expressed by using the square norm of the moment map on* $T\mathbb{R}^{2l}$ *of a certain group action, restricted to the* $2l$ *dimensional cone.*

All other cases $g \neq 4$ are homogeneous, and hence applying the moment map μ obtained in §7, we may express $F(x)$ not only by $\|\mu\|$, but also by some invariants of the moment map.

Dubrovin, Novikov, Tsarëv and Ferapontov discussed the integrability of hypersurfaces from the veiw point of Hamiltonian systems of hydrodynamic type, and studied the homogeneous case. It is interesting to investigate the non-homogeneous OT-FKM type from this point of view.

THANK YOU FOR YOUR ATTENTION.