I thank the organizers for inviting me to this conference.

I am very glad to celebrate Professor Reiko Miyaoka and Professor Keizo Yamaguchi 's 60th birthday.

Special pseudo Kähler metrics, signature, index on a deformation space of minimal surfaces in tori

Fact

The deformation space of complex structures on a Calabi-Yau manifold admits a special pseudo Kähler structure

① the period map.

② the period image is a complex Lagrangian cone

③ the period image is a Lagrangian Kähler submanifold

④ a special pseudo Kähler structure of signature $(1, P-1)$.

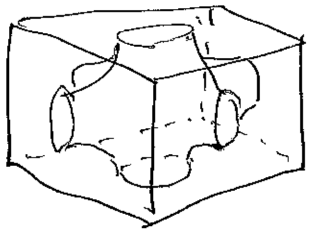
P is the dimension of the deformation space

A special pseudo Kähler structure on the deformation of a compact orientable branched minimal surface in an n-dim. (flat) torus.

① a deformation space

example

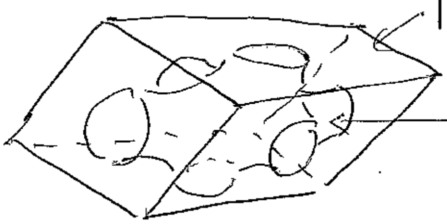
cube torus



Schwarz' P surface
(looks like a connection of jungle gym)

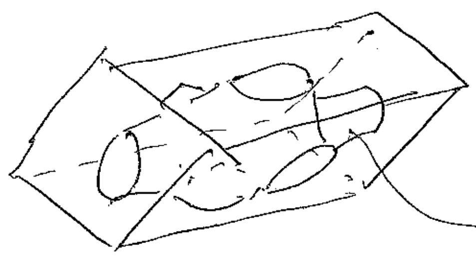
consider a linear transform

transformed cube torus



transformed Schwarz' P surface

This is not generally a minimal surface



a suitable minimal surface appears (?)

the deformation space is the set of minimal surfaces appeared under deformations of the torus

② (complex) period map

$L_{n,2r}$ = the space of $(n,2r)$ real matrices

$K_{n,r}$ = the space of (n,r) complex matrices

For $L \in L_{n,2r}$

$\langle L \rangle = \{$ column vectors of $L \}$

deformation space $\ni$

$S : (M, \{A_i, B_i\}) \longrightarrow \mathbb{R}^n / \langle L \rangle$ branched minimal immersion

$\{A_i, B_i\}$: a canonical homology basis

$\left(\int_{A_1} ds, \cdots, \int_{A_r} ds, \int_{B_1} ds, \cdots, \int_{B_r} ds\right) = L \in L_{n,2r}$

- $\langle L \rangle$ generates a lattice

complex period map

$\mapsto \left(\int_{A_1} ds^{1,0}, \cdots, \int_{A_r} ds^{1,0}, \int_{B_1} ds^{1,0}, \cdots, \int_{B_r} ds^{1,0}\right) \in K_{n,r} \times K_{n,r}$

$$\begin{tikzcd}
\text{deformation space} \ni S \arrow[r, mapsto, "\text{complex period map}"] \arrow[d, "\text{real period map}"'] & K \in K_{n,r} \times K_{n,r} \arrow[dl, "L = 2\mathrm{Re}(K)"] \\
L \in L_{n,2r} &
\end{tikzcd}$$

Question

(1) Does the complex period map give a complex Lagrangian cone ?

(2) Is the complex Lagrangian cone a Lagrangian pseudo Kähler submanifold ?

(3) Does the deformation space admit a special pseudo Kähler structure ?

Partially yes.

As an application, we give a new algorithm to compute indexa of the Jacobi operator of a minimal surface in a torus

Idea

a real generating function of a complex Lagrangian cone

1 Preliminary

$$T^*L_{n,2r} = \underset{\substack{\uparrow \\ \text{fibre}}}{L_{n,2r}} \times \underset{\substack{\uparrow \\ \text{base}}}{L_{n,2r}}$$

identification

$$\varphi : K_{n,r} \times K_{n,r} \longrightarrow L_{n,2r} \times L_{n,2r}$$

$$(Z_1, Z_2) \longmapsto \mathrm{Re}(-iZ_2, iZ_1, Z_1, Z_2)$$

$$\varphi^{-1}((L_1, L_2), (L_3, L_4)) = (L_3 - iL_2, L_4 + iL_1)$$

$\boxed{\omega = \mathrm{tr}^t dZ_2 \wedge dZ_1}$: complex symplectic form on $K_{n,r} \times K_{n,r}$

M is a submanifold in $K_{n,r} \times K_{n,r}$

Def. M is a complex Lagrangian submanifold

$\Leftrightarrow$ (1) $\dim_{\mathbb{R}} M = 2nr$ ($\Rightarrow$ M is a complex submanifol

(2) $\omega|_M = 0$

$\boxed{\eta(X, Y) = -i\omega(X, \bar{Y})}$: a Hermitian inner product of signature (nr, nr) on $K_{n,r} \times K_{n,r}$

Def. M is a Lagrangian pseudo Kähler submanifold

$\Leftrightarrow$ (1) M is a complex Lagrangian submanifold

(2) $\eta|_M$ is non-degenerate

$\boxed{n=1}$ case

$$\left(\frac{\partial\phi}{\partial L_1},\cdots,\frac{\partial\phi}{\partial L_{2r}},L_1,\cdots,L_{2r}\right)\in T^*L_{1,2r} \longleftarrow K_{1,r}\times K_{1,r}$$

$\cup$

$\boxed{\text{Lagrangian graph}}$

Re

M : Lagrangian pseudo Kähler submanifold

$\Leftrightarrow$ (if M is a complex Lagrangian submanifold)

$(L_1,\cdots,L_{2r})\in U\subset L_{1,2r}$

$Re : M\to L_{1,2r}$

the differential is surjective (bijective)

and $\boxed{\left(\frac{\partial^2\phi}{\partial L_k\partial L_\ell}\right)\in Sp(r,\mathbb{R})}\subset SL(2r,\mathbb{R})$

Locally, necessary and sufficient condition.

$$\det\left(\frac{\partial^2\phi}{\partial L_k\partial L_\ell}\right)=1.$$

If $\frac{\partial^2\phi}{\partial L_k\partial L_\ell}>0$, then T^*U admits a Ricci flat Kähler metric.

(Calabi)

$\boxed{T^*U \text{ admits a pseudo hyper Kähler metric}}$

(abstract definition : special pseudo Kähler structure (Freed)

(Cortés, Hitchin))

2. energy function

$S^2_{\mathbb{C}}$ = the space of complex symmetric matrices of size r

$RS^2_{\mathbb{C}} = \{\tau \in S^2_{\mathbb{C}} \mid \mathrm{Im}\,\tau \text{ is regular}\}$;

$H_r = \{\tau \in S^2_{\mathbb{C}} \mid \mathrm{Im}\,\tau > 0\}$: Siegel upper half space

$$\begin{array}{cccc} \Psi : & RS^2_{\mathbb{C}} \times L_{n,2r} & \longrightarrow & RS^2_{\mathbb{C}} \times K_{n,r} : \text{diffeomorphism} \\ & \Psi & & \Psi \\ & (\tau, (L_1, L_2)) & \longmapsto & (\tau, \frac{1}{2}(L_1 + i[L_1 \mathrm{Re}\,\tau - L_2](\mathrm{Im}\,\tau)^{-1})) \\ & (\tau, 2\mathrm{Re}(K, K\tau)) & \longleftarrow & (\tau, K) \end{array}$$

$$\begin{array}{cccc} \Phi : & S^2_{\mathbb{C}} \times K_{n,r} & \longrightarrow & K_{n,r} \times K_{n,r} \\ & \Psi & & \Psi \\ & (\tau, K) & \longmapsto & (K, K\tau) \end{array}$$

Def energy function on $RS^2_{\mathbb{C}} \times L_{n,2r}$

$$\boxed{E(\tau, L) = \eta(\Phi \circ \Psi(\tau, L), \Phi \circ \Psi(\tau, L))}$$

$$E(\tau, L) = \frac{1}{2} \operatorname{tr} P(\tau)\, {}^tL L$$

$$P(\tau) = \begin{pmatrix} (\operatorname{Im}\tau) + (\operatorname{Re}\tau)(\operatorname{Im}\tau)^{-1}(\operatorname{Re}\tau) & -(\operatorname{Re}\tau)(\operatorname{Im}\tau)^{-1} \\ -(\operatorname{Im}\tau)^{-1}(\operatorname{Re}\tau) & (\operatorname{Im}\tau)^{-1} \end{pmatrix}$$

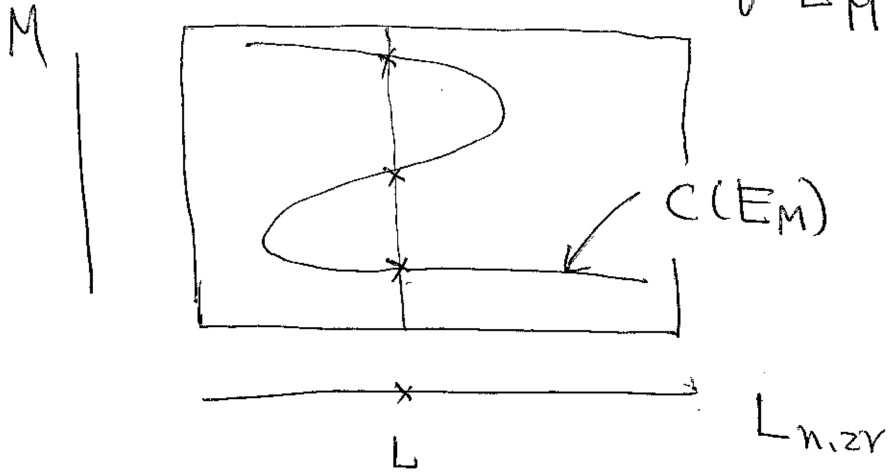
$$\in Sp(r, \mathbb{R})$$

3. Catastrophe set.

$\tau : M \subset RS^2_{\mathbb{C}}$: complex submanifold

$$E_M := E|_{M \times L_{n,2r}}$$

$$C(E_M) = \{ (\tau, L) \in M \times L_{n,2r} \mid \tau \text{ is a critical point of } E_M \text{ for a fixed } L \}$$



$$\begin{array}{ccccc} C(E_M) & \subset & M \times L_{n,2r} & \subset & RS^2_{\mathbb{C}} \times L_{n,2r} \\ \downarrow & & & & \downarrow \Psi \\ \Psi(C(E_M)) & & & \subset & RS^2_{\mathbb{C}} \times K_{n,r} \end{array}$$

$$\Psi(C(E_M)) = \left\{ (\tau, K) \in M \times K_{n,r} \;\middle|\; \operatorname{tr} \frac{\partial \tau}{\partial z^k} {}^tKK = 0,\ k = 1, \cdots \right\} \subset RS^2_{\mathbb{C}} \times K_{n,r}$$

(z^k) : M a complex coordinate system

$\boxed{\Psi(C(E_M)) \text{ is a complex manifold of } RS^2_{\mathbb{C}} \times K_{n,r} \text{ with a singularity}}$

N : an irreducible component of $C(E_M)$

$\boxed{\text{diagram}}$

$$
\begin{tikzcd}
N \subset M \times L_{n,2r} \subset RS^2_{\mathbb{C}} \times L_{n,2r} \arrow[r] \arrow[d, "\Psi"] & T^*L_{n,2r} \arrow[r, dash, "2\varphi"] \arrow[d] & K_{n,r} \times K_{n,r} \\
 & L_{n,2r} & \\
\Psi(N) \subset M \times K_{n,r} \subset RS^2_{\mathbb{C}} \times K_{n,r} \subset S^2_{\mathbb{C}} \times K_{n,r} \arrow[uurr, "\Phi"] & &
\end{tikzcd}
$$

$N \ni (\tau, L)$, $\quad T^*L_{n,2r} \ni (LP(\tau), L)$, $\quad K_{n,r} \times K_{n,r} \ni (K, K\tau)$ ← cone

$\Psi(N) \ni (\tau, K)$ $\qquad L = 2\mathrm{Re}(K, K\tau)$

ω = the complex symplectic form on $K_{n,r} \times K_{n,r}$

$\Xi = \mathrm{tr}\, d\tau\, {}^t K K$: holomorphic 1-form on $S^2_{\mathbb{C}} \times K_{n,r}$

$$\boxed{d\Xi = -2\,\Phi^*\omega}$$

$$\Xi|_{\Psi(N)} = 0 \;\Rightarrow\; (\Phi \circ \Psi)^*\omega|_N = 0$$

$\boxed{(\Phi \circ \Psi)(N) \text{ is a complex isotropic cone with a singularity}}$

N admits a non-degenerate critical point

$\Rightarrow$

N is a complex Lagrangian cone with a singularity in $T^* L_{n,2r}$

Fact

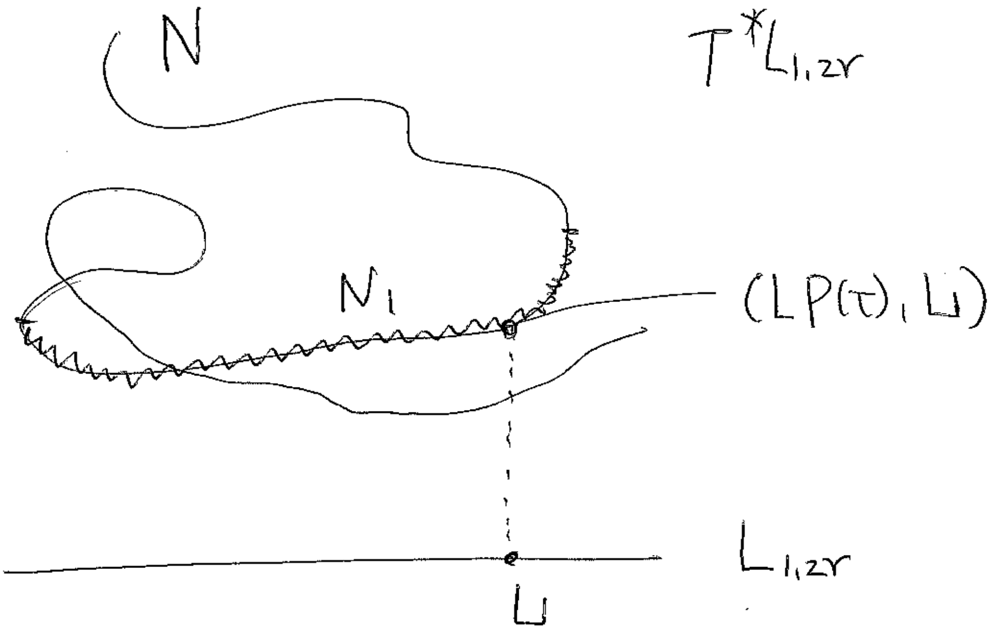
If (τ_0, L_0) is a non-degenerate critical point, then

$$\begin{array}{ccc} N & \longrightarrow & T^* L_{n,2r} \\ \cup & & \cup \\ (\tau, L) & \longmapsto & (LP(\tau), L) \end{array}$$

is a Lagrangian embedding of a neighborhood at (τ_0, L_0) (Lagrangian graph)

4. idex$_{E_M}$: index of a critical point of E_M

<u>Assumption</u> that N admits a non-degenerate critical point



N_1: a connected component of non-degenerate critical points of N

$\boxed{\text{$N_1$ admits a special pseudo Kähler structure with signature (p,q)}}$

index$_{E_M}$, signature are invariants of N_1

To compute index_{E_M}

formula

$$\sum_{a,b} \mathrm{Hess}\, E_M\left(\frac{\partial}{\partial \tau^a}, \frac{\partial}{\partial \tau^b}\right) \frac{\partial \tau^a}{\partial L_k} \frac{\partial \tau^b}{\partial L_l}$$

$$= P(\tau(L))_{kl} - \mathrm{Hess}\, a\left(\frac{\partial}{\partial L_k}, \frac{\partial}{\partial L_l}\right)$$

where $a(L) = E_M(\tau(L), L)$

the surjective condition at L_0

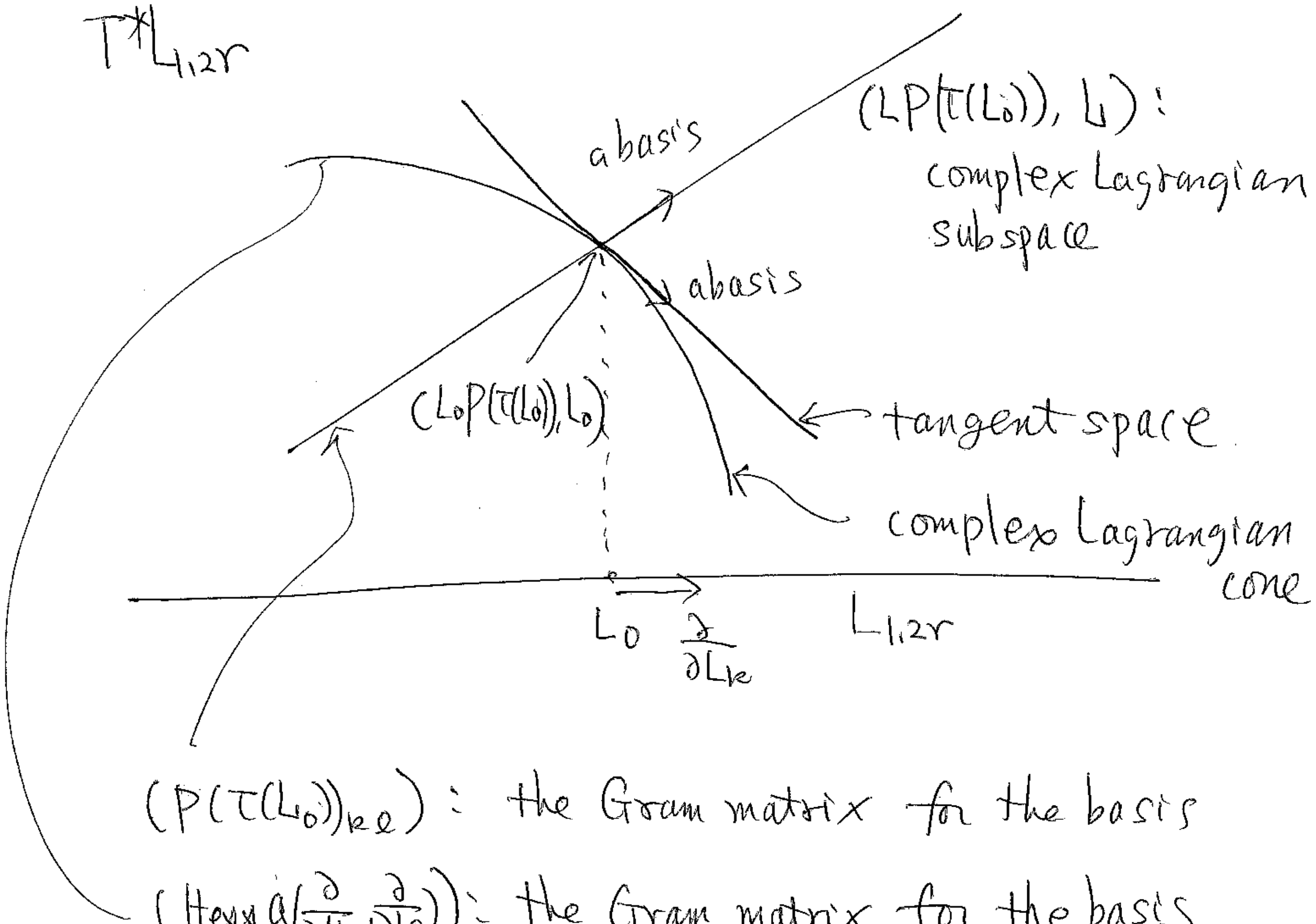
the differential of $L \to \tau(L)$ is surjective at L_0

If the surjective condition holds at $(\tau(L_0), L_0)$, then

$\mathrm{index}_{E_M} =$ the number of negative eigenvalues of $\left(P(\tau(L_0))_{kl}\right) - \left(\mathrm{Hess}\, a\left(\frac{\partial}{\partial L_k}, \frac{\partial}{\partial L_l}\right)\right)$

$$\Rightarrow \quad \boxed{2g + \mathrm{index}_{E_M} \leq 2\dim_{\mathbb{C}} M}$$

A geometric meaning $(P(\tau(L_0))_{k\ell}), (\mathrm{Hess}\, a(\frac{\partial}{\partial L_k}, \frac{\partial}{\partial L_\ell}))$



$(P(\tau(L_0))_{k\ell})$: the Gram matrix for the basis

$(\mathrm{Hess}\, a(\frac{\partial}{\partial L_k}, \frac{\partial}{\partial L_\ell}))$: the Gram matrix for the basis

Idea

Using the tangent space of the complex Lagrangian cone in $K_{1,r} \times K_{1,r}$, we compute two Gram matrices

$\boxed{\text{Algorithm}}$

$\{T_1, \cdots, T_r\}$: a basis of the tangent space of the complex Lagrangian cone in $K_{n,r} \times K_{n,r}$

(1) Check $\det(\eta(T_k, T_\ell)) \neq 0$

$\Rightarrow$ non-degenerate critical point

(we can obtain the signature)

(2) $T_k = (A_k, B_k) \quad \in K_{1,r} \times K_{1,r}$
$T_k' = (iA_k, iB_k)$

Set $(C_k, D_k) = 2\mathrm{Re}(A_k, B_k)$, $(C_k', D_k') = 2\mathrm{Re}(iA_k, iB_k)$

$\Rightarrow \{(C_k, D_k), (C_k', D_k')\}$ is a basis of $L_{1,2r}$

(3) Make a basis of $K_{1,r} \times K_{1,r}$ from this basis and calculate two Gram matrix.

$E_k = (\mathrm{Re}\, A_k) + i\,[(\mathrm{Re}\, A_k)\,\mathrm{Re}\,\tau - (\mathrm{Re}\, B_k)]\,(\mathrm{Im}\,\tau)^{-1}$
$E_k' = (\mathrm{Re}\, iA_k) + i\,[(\mathrm{Re}\, iA_k)\,\mathrm{Re}\,\tau - (\mathrm{Re}\, iB_k)]\,(\mathrm{Im}\,\tau)^{-1}$

$\{(E_k, E_k\tau), (E_k', E_k'\tau)\}$: a real basis of complex Lagrangian subspace for P

$\{T_k, T_k'\}$: a real basis of the tangent space

W_2: the Gram matrix for $\{(E_k, E_k\tau), (E'_k, E'_k\tau)\}$

W_1: the Gram matrix for $\{T_k, T'_k\}$

for $2\mathrm{Re}\,\eta$

$$\boxed{\text{index}_{E_M} = \text{the number of negative eigenvalues of } W_2 - W_1}$$

The tangent space at a point of N_1 in $T^* L_{12r}$ determines the signature and index_{E_M} (under the surjective condition)

5. null submanifold

Lag^C = the space of complex Lagrangian subspaces in $K_{1,r} \times K_{1,r}$

Lag_1^C = the space of non-degenerate complex Lagrangian subspaces in $K_{1,r} \times K_{1,r}$

$$RS_C^2 \subset Lag_1^C \subset Lag^C$$

$\tau \in RS_C^2$ $\rightsquigarrow$ $\{(k, k\tau) \mid k \in K_{1,r}\} \in Lag_1^C$

corresponding ~~corresponding~~ nondegenerate complex Lagrangian subspace

$M \subset RS_C^2$ $\longrightarrow$ $T \subset K_{1,r} \times K_{1,r}$

a complex submanifold ($\tau \in M$) — complex Lagrangian cone (non-degenerate) ($(k, k\tau) \in T$)

G : Gauss map

$G(T) \subset Lag_1^C$, $G((k, k\tau)) \in G(T)$

G(T) is a null submanifold

(a generalization of a null curve in Q^3)

$r=2$, $Lag^C = Q^3$ Bryant

$$\boxed{\text{Hess } E_M \equiv 0 \iff M \text{ is a null submanifold}}$$

M itself is the Gauss image

6. Application

M: a Riemann surface of genus r

$\{A_i, B_i\}$: a canonical homology basis

$\{\varphi^k\}$: dual 1-forms (holomorphic)

$$\int_{A_i} \varphi^j = \delta_{ij}$$

then

$$(\tau_{ij}) = \left(\int_{B_i} \varphi^i\right) \in H_r \subset RS^2_{\mathbb{C}}$$

$\uparrow$

Riemann matrix

RM = the space of Riemann matrices

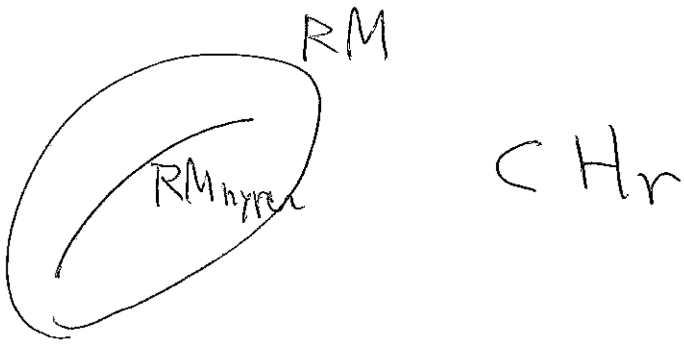
$RM_{\text{non-hyper}} = \{\tau \in RM \mid M \text{ is non-hyperelliptic}\}$

$RM_{\text{hyper}} = \{\tau \in RM \mid M \text{ is hyperelliptic}\}$

then $\dim RM_{\text{non-hyper}} = 3r-3$

$\dim RM_{\text{hyper}} = 2r-1$

RM_{hyper} is a singularity set of RM



$C(E_{RM_{\text{non-hyper}}})$, $C(E_{RM_{\text{hyper}}})$

$(\tau, L) \in C(E_{RM_{\text{non-hyper}}})$, $C(E_{RM_{\text{hyper}}})$

$L = (L_1, L_2)$

M : Riemann surface

$S : M \longrightarrow \mathbb{R}^n$ by

$$\boxed{\int_{P_0}^{P} (L_1, L_2)\, T_\tau^{-1} \begin{pmatrix} \mathrm{Re}\,\psi_1 \\ \vdots \\ \mathrm{Re}\,\psi_r \\ \mathrm{Im}\,\psi_1 \\ \vdots \\ \mathrm{Im}\,\psi_r \end{pmatrix}, \quad T_\tau = \begin{pmatrix} E_r & \mathrm{Re}\,\tau \\ 0 & \mathrm{Im}\,\tau \end{pmatrix}}$$

S is a multivalued weakly conformal harmonic map s.t.

$$\left(\int_{A_1} dS, \ldots, \int_{A_r} dS, \int_{B_1} dS, \ldots, \int_{B_r} dS\right) = (L_1, L_2)$$

If $\langle L \rangle$ generates a lattice, then

$S : M \longrightarrow \mathbb{R}^n / \langle L \rangle$: branched minimal surface

$$ds^{1,0} = K\begin{pmatrix}\psi_1\\ \vdots \\ \psi_r\end{pmatrix}$$

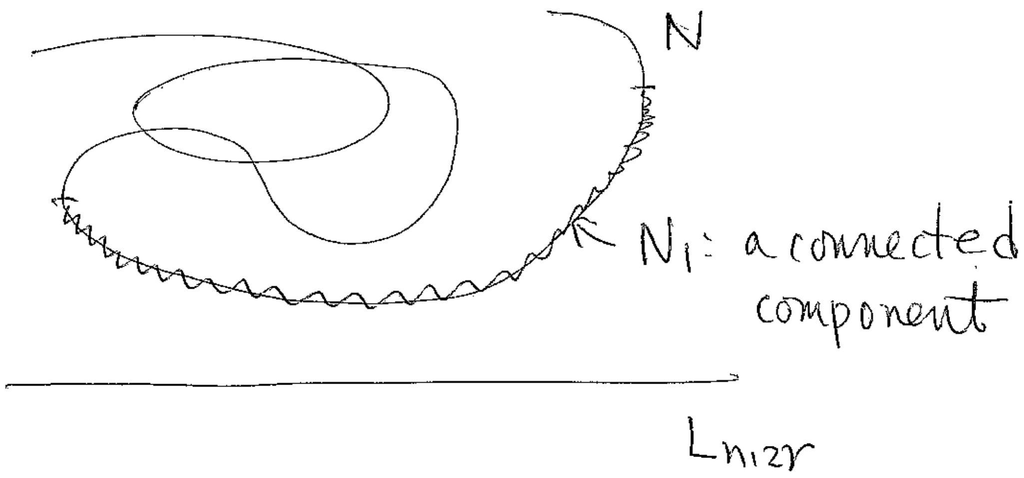
$$K = \frac{1}{2}\left(L_1 + i\,[L_1 \mathrm{Re}\,\tau - L_2](\mathrm{Im}\,\tau)^{-1}\right)$$

and

$$\left(\int_{A_1} ds^{1,0}, \dots, \int_{A_r} ds^{1,0}, \int_{B_1} ds^{1,0}, \dots, \int_{B_r} ds^{1,0}\right) = (K, K\tau)$$

N : an irreducible component of
$C(E_{RM_{\text{non-hyper}}})$, $C(E_{RM_{\text{hyper}}})$

Assume N admits a non-degenerate critical point.



N_1 admits a special pseudo Kähler structure of signature (p, q).

$T_{n,2r} = \{ L \in L_{n,2r} \mid \langle L \rangle \text{ is a lattice} \} \subset L_{n,2r}$

$\boxed{\text{dense}}$

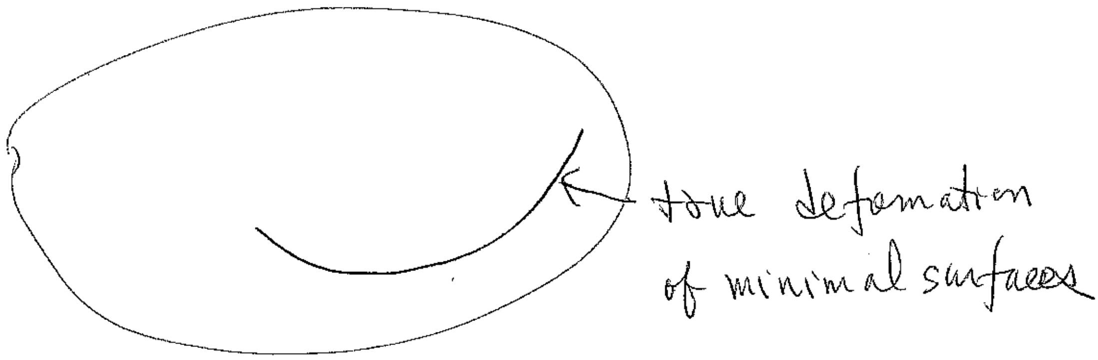
So $\exists$ dense set of N gives branched minimal surfaces.

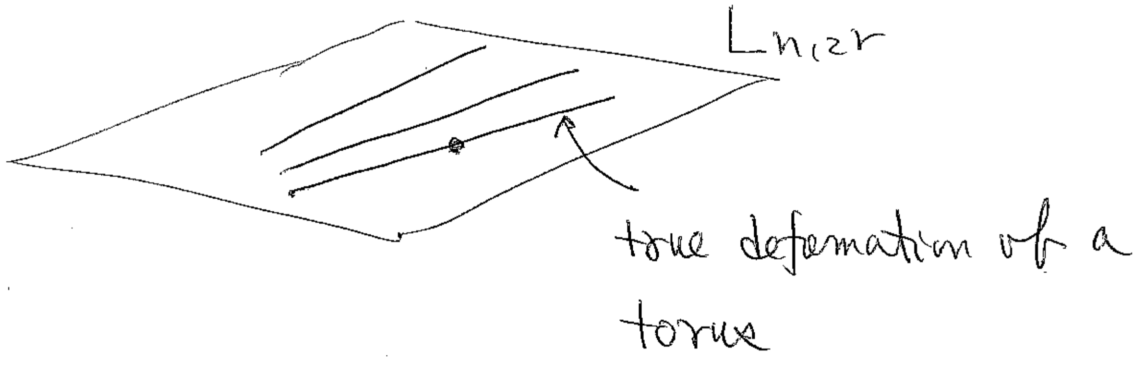
We may consider that

$C(E_{M\,\text{non-hyper}}),\quad C(E_{M\,\text{hyper}})$

are deformations of minimal surfaces.

Note





Ejiri

(1) $(\tau, L) \in N_1 \subset C(RM_{\text{non-hyper}})$

$\wr$

corresponding minimal surface

$$\boxed{\begin{array}{l} \text{index}_a = \text{index}\, E_{RM_{\text{non-hyper}}} \\ \text{nullity}_a = n \end{array}}$$

(2) $(\tau, L) \in N_1 \subset C(RM_{\text{hyper}})$

$\wr$

corresponding minimal surface
and <u>immersed</u>

$$\boxed{\begin{array}{l} \text{index}_a = \text{index}\, E_{RM_{\text{hyper}}} + \alpha \\ \text{nullity}_a = n + 2r - 4 - 2\alpha \\ (0 \leqq \alpha \leqq r-2) \end{array}}$$

Algorithm of index$_a$ for a hyperelliptic minimal surface of genus 3 in T^3

a minimal surface of genus 3 in T^3

a hyperelliptic Riemann surface

$$y^2 = (z-a_1)\cdots(z-a_8)$$

$$w_1 = \frac{1-z^2}{y}dz,\quad w_2 = \frac{i(1+z^2)}{y}dz,\quad w_3 = \frac{2z}{y}dz$$

$\{A_i, B_i\}$: a canonical homology basis

$$\mathrm{Re}\left(\int_{A_i}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}, \int_{B_i}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}\right) = (L_1, L_2)$$

$\langle L \rangle$ gives a lattice

A neighborhood of the minimal surface

$$\boxed{\begin{array}{l} \left(\int_{A_i} \alpha g \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \int_{B_i} \alpha g \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \in K_{3,3} \times K_{3,3} \\ \alpha \in \mathbb{C}^*, \quad g \in SO(3,\mathbb{C}) \\ a_1, \cdots, a_5 : \text{variable } (a_6, a_7, a_8 : \text{fix}) \end{array}}$$

$\boxed{\text{the surjective condition holds}}$

Calculation 1

$$T_i = \frac{\partial}{\partial a_i} \left(\int_{A_i} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \int_{B_i} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \quad i = 1, \cdots, 5$$

$$T_6 = \left(\int_{A_i} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \int_{B_i} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) = (C_1, C_2)$$

$$T_7 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_6, \quad T_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} T_6, \quad T_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} T_6$$

$\tau = C_1^{-1} C_2$: Riemann matrix

Calculation 2

$$\det\left(\eta(T_i, T_j)\right) \neq 0$$

$\Rightarrow$ signature (p, q)

$\exists N_1 \ni$ the minimal surface

Calculation 3

algorithm

$$\text{index}_{ERM_{hyper}} = \text{index}\,(W_2 - W_1)$$

$$\text{index}_a = \text{index}_{ERM_{hyper}} + 1$$

($\alpha = 1 \Leftarrow$ a Micallef's result)

Now, joint work with Shoda

(1) P surface	$\mathrm{index}_a = 1$ (Ross)	$(p, q) = (4, 5)$
(2) CLP surface	$\mathrm{index}_a = 3$ (Montiel-Ros, Nayatani)	$(p, q) = (6, 3)$
(3) A deformation of P surface to CLP surface	$\mathrm{index}_a = 2$	$(p, q) = (5, 4)$
(4) A deformation of H surface	$\mathrm{index}_a = 1$	$(p, q) = (4, 5)$
	$\mathrm{index}_a = 2$	$(p, q) = (5, 4)$
	$\mathrm{index}_a = 3$	$(p, q) = (6, 3)$
(5) A deformation of TT surface	$\mathrm{index}_a = 1$	$(p, q) = (4, 5)$
	$\mathrm{index}_a = 2$	$(p, q) = (5, 4)$