Isometric immersions of the hyperbolic plane into the hyperbolic space

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Differential Geometry and Tanaka Theory

$\sim$ Differential System and Hypersurface Theory $\sim$

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1.1 Negative & Non-negative curvature

Consider isometric immersions between space forms of same curvature with codim. $= 1$.

- $\triangledown$ $f : \boldsymbol{R}^n \xrightarrow{\text{isom. imms.}} \boldsymbol{R}^{n+1} \iff$ Cylinder over plane curve (Hartman-Nirenberg 1959, Massey 1962 ($n = 2$))
- $\triangledown$ $f : \boldsymbol{S}^n \xrightarrow{\text{isom. imms.}} \boldsymbol{S}^{n+1} \iff$ Totally geodesic (O'Neil-Stiel 1963)
- $\triangledown$ $f : \boldsymbol{H}^n \xrightarrow{\text{isom. imms.}} \boldsymbol{H}^{n+1} \rightsquigarrow \exists$ **Non-Trivial Examples** (Nomizu 1973, Abe-Haas 1990)

1.2 Developable Surfaces

Fact

Isom. Imms. $f : \boldsymbol{H}^2 \to \boldsymbol{H}^3$

$\Longleftrightarrow$ Complete **Developable** Surfaces in $\boldsymbol{H}^3$.

- **Developable** $\overset{\text{def}}{\Longleftrightarrow}$ Extrinsically Flat & Ruled.
- Extrinsically Flat $\overset{\text{def}}{\Longleftrightarrow}$ product of principal curv. $\equiv 0$.
- Ruled $\overset{\text{def}}{\Longleftrightarrow}$ Locus of a motion of geodesics.

- Gauss equation

$$\lambda_1 \lambda_2 = K + 1, \qquad (\lambda_1, \lambda_2 : \text{ principal curv.})$$

- Proof : an analogue of the method used by Massey for the Euclidean case.

$\mathcal{L}(\boldsymbol{H}^3)$: the Space of Oriented Geodesics in $\boldsymbol{H}^3$.

$$\text{Ruled Surfaces in } \boldsymbol{H}^3 \overset{\text{corresp.}}{\longleftrightarrow} \text{Curves in } \mathcal{L}(\boldsymbol{H}^3).$$

$$\textbf{Developable} \text{ Surfaces in } \boldsymbol{H}^3 \overset{\text{corresp.}}{\longleftrightarrow} \underline{\textbf{???}} \text{ Curves in } \mathcal{L}(\boldsymbol{H}^3).$$

Theorem I (H)

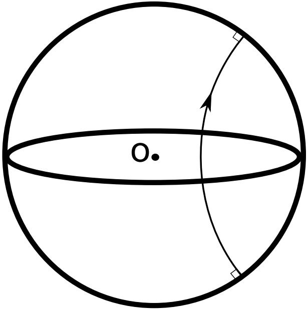
$$\text{Developables} \overset{\text{corresp.}}{\longleftrightarrow} \text{Curves in } \mathcal{L}(\boldsymbol{H}^3) \text{ s.t. } \begin{cases} \textbf{null} \text{ w.r.t. } G \\ \textbf{causal} \text{ w.r.t. } \hat{G}. \end{cases}$$

- G, $\hat{G}$: certain neutral metrics on $\mathcal{L}(\boldsymbol{H}^3)$.
- A curve $\alpha : \boldsymbol{R} \longrightarrow (\mathcal{L}(\boldsymbol{H}^3), \langle\ ,\ \rangle)$: **null** (resp. **causal**) $\overset{\text{def}}{\Longleftrightarrow} \langle \alpha', \alpha' \rangle = 0$ (resp. $\langle \alpha', \alpha' \rangle \leq 0$).

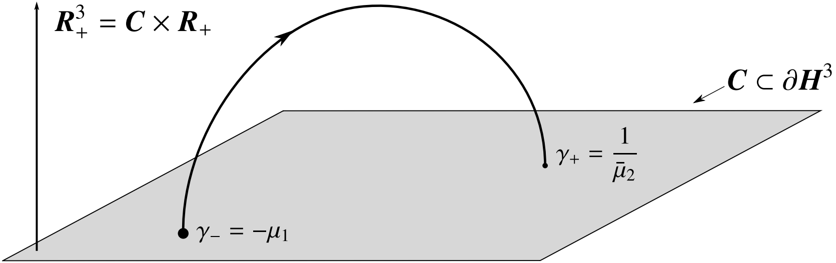
2 The space of oriented geodesics in H^3

$$\mathcal{L}(\boldsymbol{H}^3) := \left\{ [\gamma] \,|\, \gamma\text{: an unit speed geodesic in } \boldsymbol{H}^3 \right\}$$

(where, $\gamma_1 \sim \gamma_2 \overset{\text{def}}{\iff} \exists T;\ \gamma_1(T + \cdot) = \gamma_2(\cdot)$).



$$\mathcal{L}(\boldsymbol{H}^3) = \boldsymbol{S}^2 \times \boldsymbol{S}^2 \setminus \{\text{Diagonal.}\}$$



- (μ_1, μ_2) : a (complex) coordinate system of $\mathcal{L}(\boldsymbol{H}^3)$.

$$\mathcal{L}(\boldsymbol{H}^3) = (\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \Delta, \qquad (\Delta = \{1 + \mu_1\bar{\mu}_2 = 0\}).$$

- Set

$$G := \operatorname{Im}\left[\frac{4d\mu_1 d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2}\right], \qquad \hat{G} := \operatorname{Re}\left[\frac{4d\mu_1 d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2}\right],$$

: **metrics** on $\mathcal{L}(\boldsymbol{H}^3)$ $\quad$ $(\mathrm{sgn} = (++--)$: neutral).

2.1 What are G and $\hat{G}$?

G, $\hat{G}$: metrics invariant under $\mathrm{Isom}_0(\boldsymbol{H}^3) \curvearrowright \mathcal{L}(\boldsymbol{H}^3)$.
(=:**Invariant Metrics**)

Fact (Salvai 2007)

Any invariant metrics on $\mathcal{L}(\boldsymbol{H}^3)$ can be written as

$$aG + b\hat{G} \qquad (a, b \in \boldsymbol{R}).$$

What are specific properties of G, $\hat{G}$?

Geometric structures ω, J, P on $\mathcal{L}(\boldsymbol{H}^3)$

▽ [Canonical Symplectic Structure ω]

$$\omega := \hat{\pi}_* (d\Theta) .$$

- $\hat{\pi} : U\boldsymbol{H}^3 \ni (p, v) \longmapsto [\gamma_{p,v}] \in \mathcal{L}(\boldsymbol{H}^3)$: geodesic flow.
- Θ: The canonical contact form of $U\boldsymbol{H}^3$ (Liouville form).

▽ [Minitwistor Complex Structure J (Hitchin 1982)]

- $T_{[\gamma]}\mathcal{L}(\boldsymbol{H}^3) = \mathcal{J}^{\perp}(\gamma) :=$ {orthogonal Jacobi field along γ}.
- $J_{[\gamma]} : \mathcal{J}^{\perp}(\gamma) \ni V \longmapsto \gamma' \times_\gamma V \in \mathcal{J}^{\perp}(\gamma)$: rotation by $90°$, (where $\times$: vector product of $\boldsymbol{H}^3$).

▽ [para-Complex Structure P (Kaneyuki-Kozai, et al.)]

$P_{[\gamma]} : T_{[\gamma]}\mathcal{L}(\boldsymbol{H}^3) \to T_{[\gamma]}\mathcal{L}(\boldsymbol{H}^3)$;

$$\frac{\partial}{\partial \mu_1} \mapsto -\frac{\partial}{\partial \mu_1}, \qquad \frac{\partial}{\partial \mu_2} \mapsto \frac{\partial}{\partial \mu_2}.$$

Characterizations of G, $\hat{G}$

Proposition (H)

Let ω, J, P as above, then

$$G = 2\omega(J\cdot, \cdot), \qquad \hat{G} = 2\omega(P\cdot, \cdot).$$

(Other characterizations)

- Conformally flat invariant metrics $\iff$ λG ($\lambda \in \boldsymbol{R}$)
- Einstein invariant metrics $\iff$ $\lambda \hat{G}$ ($\lambda \in \boldsymbol{R}$)

Some remarks on $\mathcal{G} = G + \sqrt{-1}\hat{G} = \frac{4d\mu_1 d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2}$

▽ $-\mathcal{G}$: (complex) metric on $\mathcal{L}(\boldsymbol{H}^3) = \mathrm{SL}(2,\boldsymbol{C})/\mathrm{GL}(1,\boldsymbol{C})$.

▽ $(\mathcal{L}(\boldsymbol{H}^3), \mathcal{G}) = (\mathbb{P}^1, g_{FS})^{\boldsymbol{C}}$: complexification.

Remark: General case Σ^3 : a 3-dim. space form.

$$G := 2\omega(J\cdot, \cdot)$$

: a neutral metric on $\mathcal{L}(\Sigma^3)$.

▽ [Canonical Symplectic Structure ω]

$$\omega := \hat{\pi}_*(d\Theta).$$

- $\hat{\pi} : U\Sigma^3 \ni (p, v) \longmapsto [\gamma_{p,v}] \in \mathcal{L}(\Sigma^3)$: geodesic flow.
- Θ: The canonical contact form of $U\Sigma^3$ (Liouville form).

▽ [Minitwistor Complex Structure J (Hitchin 1982)]

- $T_{[\gamma]}\mathcal{L}(\Sigma^3) = \mathcal{J}^\perp(\gamma) :=$ {orthogonal Jacobi field along γ}.
- $J_{[\gamma]} : \mathcal{J}^\perp(\gamma) \ni V \longmapsto \gamma' \times_\gamma V \in \mathcal{J}^\perp(\gamma)$: rotation by 90°.
- J is integrable if Σ is a space form .

Proposition (H)

Developable surfaces in $\Sigma^3 \Longrightarrow$ Null curves in $(\mathcal{L}(\Sigma^3), G)$.

2.2 Representation Formula

Theorem I (H)

Developable surfaces generated by complete geodesics

$$\overset{\text{corresp.}}{\longleftrightarrow}$$

Curves in $\mathcal{L}(\boldsymbol{H}^3)$ s.t. null w.r.t. G and causal w.r.t. $\hat{G}$.

- Nullity for G = Extrinsically Flatness.
- Causality for $\hat{G}$ = Regularity (of Surface).

▽ [Representation formula for developables]

A curve $\alpha = (\mu_1(s), \mu_2(s)) : \boldsymbol{R} \to \mathcal{L}(\boldsymbol{H}^3) \overset{\text{bihol}}{\cong} (\hat{C} \times \hat{C}) \setminus \Delta$:

s.t. $$\begin{cases} \operatorname{Im} \frac{4\mu_1'(s)\bar{\mu}_2'(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} = 0 & \text{(i.e., } G(\alpha', \alpha') = 0\text{),} \\ \operatorname{Re} \frac{4\mu_1'(s)\bar{\mu}_2'(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} \leq 0 & \text{(i.e., } \hat{G}(\alpha', \alpha') \leq 0\text{).} \end{cases}$$

$$\Rightarrow \qquad f(s,t) = \frac{1}{2|1+\mu_1\bar{\mu}_2|} \begin{pmatrix} (|\mu_2(s)|^2 + 1)e^t + (|\mu_1(s)|^2 + 1)e^{-t} \\ 2(e^t \operatorname{Re}\mu_2(s) - e^{-t}\operatorname{Re}\mu_1(s)) \\ 2(e^t \operatorname{Im}\mu_2(s) - e^{-t}\operatorname{Im}\mu_1(s)) \\ -(|\mu_2(s)|^2 - 1)e^t + (|\mu_1(s)|^2 - 1)e^{-t} \end{pmatrix} \in \boldsymbol{H}^3$$

gives a developable surface, where

$\boldsymbol{H}^3 = \left\{ \boldsymbol{x} = {}^t(x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4_1 \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 > 0 \right\}$,

($\boldsymbol{R}^4_1$: the Lorentz-Minkowski 4-space).

(Ex. 1) [Totally Geodesic]

- $\mu_1(s) = -\tanh s, \qquad \mu_2(s) = \tanh s.$
- $\mathcal{G}(\alpha', \alpha') = \frac{4\mu_1'(s)\bar{\mu}_2'(s)}{(1+\mu_1(s)\bar{\mu}_2(s))^2} = -4.$

  $\therefore \quad G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') < 0.$

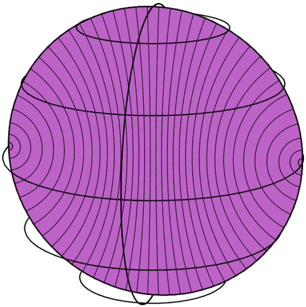


Figure: Totally geodesic.

(Ex. 2) [Hyperbolic analogues of Cylinders]

- $$\mu_1(s) = -\zeta(s), \qquad \mu_2(s) = \zeta(s)$$
  (where, $\zeta(s) : \boldsymbol{R} \longrightarrow \boldsymbol{D} \subset \boldsymbol{C}$: regular curve).
- $$\mathcal{G}(\alpha', \alpha') = -\frac{4|\zeta'(s)|^2}{(1+|\zeta(s)|^2)^2} < 0.$$
  $$\therefore \quad G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') < 0.$$

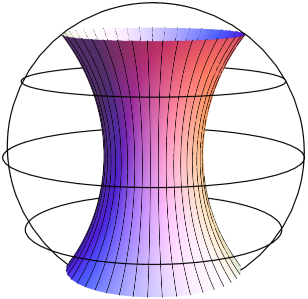


Figure: $\zeta(s) = e^{is}/3$.

(Ex. 3) [Ideal Cones]

- $\mu_1(s) = \text{const.}, \qquad \mu_2(s) = \mu(s)$
  (where, $\mu(s) : \boldsymbol{R} \longrightarrow \boldsymbol{C}$: regular curve).
- $\mathcal{G}(\alpha', \alpha') = 0.$
  $\therefore \quad G(\alpha', \alpha') = 0 \quad \& \quad \hat{G}(\alpha', \alpha') = 0.$

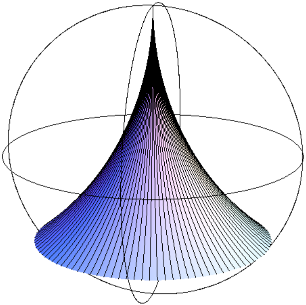


Figure: const. $= 0$, $\mu(s) = e^{is}/2$.

(Ex. 4) [Rectifying Developables of Helices]

$$\mu_1(s) := \kappa \frac{4\sqrt{2}\sqrt{\kappa^2+\tau^2}i + 4\tau A_-}{(\sqrt{2}\sqrt{\kappa^2+\tau^2}i + 4\tau A_+)(a_+ + a_-)^2 + 4\kappa A_-} \exp\left(\frac{A_+ + iA_-}{\sqrt{2}} s\right),$$

$$\mu_2(s) := \frac{1}{\kappa} \frac{(\sqrt{2}\sqrt{\kappa^2+\tau^2} - \tau A_+)(a_+ + a_-)^2 - 4\kappa A_-}{4\sqrt{2}\sqrt{\kappa^2+\tau^2}i + 4\tau A_- - (a_+ + a_-)^2 A_+} \exp\left(\frac{-A_+ + iA_-}{\sqrt{2}} s\right)$$

(where, $\kappa,\ \tau \in \boldsymbol{R} \setminus \{0\}$,

$$a_\pm := \sqrt{(\kappa \pm 1)^2 + \tau^2},\ A_\pm := \sqrt{\pm(1-\kappa^2-\tau^2) + a_+ a_-}).$$

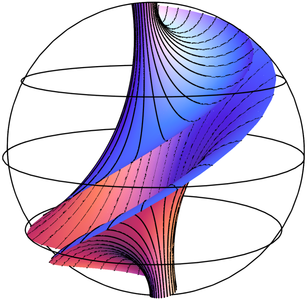


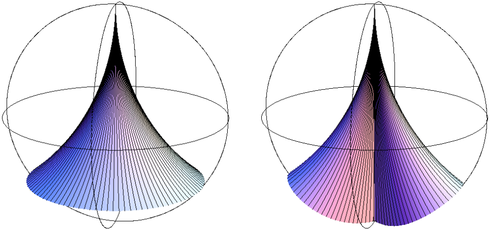
Figure: $\kappa = \tau = 1$.

3.1 Behavior at infinity: Ideal Cones

Proposition

Complete developables corresp. to curves null w.r.t. G, $\hat{G}$ have an end asymptotic to a point in $\partial \boldsymbol{H}^3$.

Complete developables corresp. to curves null w.r.t. G, $\hat{G}$
$\overset{\text{def}}{\Longleftrightarrow}$: **Ideal Cones.**



3.2 Developables of Exponential Type

Lem (Hyperbolic Massey's lemma)

$f : M^2 \to \boldsymbol{H}^3$: extrinsically flat surface.

H: mean curvature.

l: an asymptotic curve in non umbilic point set.

t: arc-lenghth parameter of l.

$$\Rightarrow \qquad \frac{\partial^2}{\partial t^2}\left(\frac{1}{H}\right) = \frac{1}{H} \qquad \text{on } l.$$

▽ Thus, $\dfrac{1}{H} = P\cosh t + Q\sinh t$:

$$\frac{1}{H} = \begin{cases} A\cosh(t+B) \\ Ae^{\pm t} \\ A\sinh(t+B) \end{cases}$$

▽ For a **complete** developables, $1/H$ never vanishes.
$\Longrightarrow$ third case does not occur. Thus,

(c) $H = a(s)/\cosh(t + b(s))$ or

(e) $H = \rho(s)e^t$ holds.

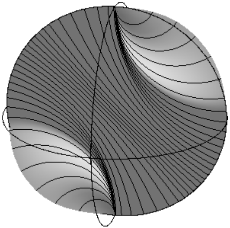
▽ Complete dev. with (e) $\overset{\text{def}}{\Longleftrightarrow}$: **Exponential Type.**

▽ Example: Ideal cones $\Rightarrow$ Exponential Type.
(Converse?)

Theorem II (H)

Real analytic developable surfaces of exponential type are ideal cones.

▽ Rem. ∃ non-real-analytic exponential dev. which is not an ideal cone.



(Review.)

- ▽ Lem 1. Two unit speed geodesics in $\boldsymbol{H}^3$

$$\alpha(t) = (\cosh t)p + (\sinh t)v, \quad \beta(t) = (\cosh t)q + (\sinh t)w$$

are asymptotic if and only if

$$\langle p + v, q + w \rangle = 0.$$

- ▽ Lem 2.(Frenet-Serret formula) Let $\mathcal{F} = (\gamma, \boldsymbol{e} = \gamma', \boldsymbol{n}, \boldsymbol{b})$ be the Frenet frame for a curve γ in $\boldsymbol{H}^3$

$$\Rightarrow \quad \gamma'' = \gamma + \kappa \boldsymbol{n}, \quad \boldsymbol{n}' = -\kappa \boldsymbol{e} + \tau \boldsymbol{b}, \quad \boldsymbol{b}' = -\tau \boldsymbol{n}.$$

Sketch of the proof

▽ Let f be a real analytic exponential developable

$$f(s,t) = (\cosh t)\gamma(s) + (\sinh t)\xi(s) \qquad \left(\in \boldsymbol{H}^3 \subset \boldsymbol{R}^4_1\right),$$

such that $H(s,t) = \delta(s)e^t$.

▽ By Lem 1, it suffices to prove that

$$(\varphi(s) :=) \, \langle \gamma(s) + \xi(s), \gamma(s_0) + \xi(s_0) \rangle \equiv 0 \qquad (1)$$

for some $s_0 \in \boldsymbol{R}$.

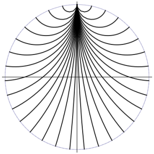
▽ To show (1), we shall determine ξ.

(To determine ξ)

1 [Determine the geodesic foliation of $\boldsymbol{H}^2$ induced by f]

$F(s,t) = (\cosh t)c(s) + (\sinh t)v(s)$: Geod. Foli. of $\boldsymbol{H}^2$

Codazzi equation ⇒ c(s) : horocycle.



2 [Represent ξ in terms of $\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}$] **Gauss equation** ⇒

$$\xi = f_*(v(s)) = \frac{\boldsymbol{n}(s) + \delta(s)\boldsymbol{b}(s)}{\kappa(s)}.$$

Finally, applying Lem2 (Frenet-Serret), it holds that

$$\varphi'(s) = \left\langle \gamma(s) + \frac{\boldsymbol{n}(s) + \delta(s)\boldsymbol{b}(s)}{\kappa(s)}, \gamma(s_0) + \xi(s_0) \right\rangle' \equiv 0. \qquad \square$$