Geometric characterization of Monge-Ampère equations

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I'll talk about characterization of Monge-Ampère equation

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0, \qquad (1)$$

where each capital letter indicates a function of variables x, y, z, z_x, z_y. A single second order PDE of one unknown function with two independent variables corresponds to a hypersurface R in 2-jet space $J^2(\mathbb{R}^2, \mathbb{R})$ with coordinates (x, y, z, p, q, r, s, t) ($p := z_x, q := z_y, r := z_{xx}, s := z_{xy}, t := z_{yy}$). On the other hand, it is well-known that a Monge-Ampère equation can be expressed in terms of exterior differential system (EDS)—*Monge-Ampère system* $\mathcal{I}$. I studied the relation between *Monge characteristic systems* of Monge-Ampère equation R and *those* of Monge-Ampère system $\mathcal{I}$. From this observation, I characterized Monge-Ampère equations as hypersurfaces R in $J^2(\mathbb{R}^2, \mathbb{R})$.

Lagrange-Grassmann bundle and single second-order PDEs

Let (J, C) be a contact manifold, i.e. C is a differential system of corank 1 on a manifold J such that C is locally defined by a 1-form θ satisfying $\theta \wedge (d\theta)^n \neq 0$ at each point.

Lagrange-Grassmann bundle $L(J)$ (geometric 2-jet space) over (J, C)

$$L(J) := \bigcup_{u \in J} L(J)_u \xrightarrow{\quad \pi \quad} J$$

where $L(J)_u$ is the Grassmannian of all Lagrangian subspaces of the symplectic vector space $(C(u), d\theta_u)$.

The *canonical system* $E \subset T(L(J))$ is defined by

$$E(v) := \pi_*^{-1}(v) \subset T_v(L(J)) \quad \text{for} \quad v \in L(J).$$

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The *canonical system* $E \subset T(L(J))$ is defined by

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We consider a single second order PDE (R, D) of one unknown function with two independent variables: Set the dimension of J is 5.

$$R \subset L(J) : \text{a hypersurface such that } \pi|_R \text{ is submersion}$$
$$D = E|_R : \text{the restriction of } E \text{ to } R$$

Let us fix a point $v_o \in R$. It is well-known that the structure equation of $D = \{ \varpi_0 = \varpi_1 = \varpi_2 = 0 \}$ is expressed as follows:
If the equation R is *hyperbolic* around v_o, the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \qquad\qquad\quad \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ is a coframe around $v_o \in R$.

Then the *Monge characteristic systems* $\mathcal{M}_i$ of (R, D) are defined as

$$\mathcal{M}_1 = \{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0 \}$$

and

$$\mathcal{M}_2 = \{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{22} = 0 \}.$$

If the equation R is *parabolic* around v_o, the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad\quad \omega^2 \wedge \pi_{12} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0,\ \varpi_1,\ \varpi_2,\ \omega^1,\ \omega^2,\ \pi_{12},\ \pi_{22}\}$ is a coframe around $v_o \in R$.

Then the *Monge characteristic system* $\mathcal{M}$ of (R, D) is defined as

$$\mathcal{M} = \{\, \varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0 \,\}.$$

notation

Let D be a differential system on a manifold Σ. The *first derived system* ∂D of D is defined by, in terms of sections,

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$$

where $\mathcal{D}$ is the space of sections of D and $[\ ,\ ]$ is Lie bracket for vector fields. Furthermore, the *k-th derived system* $\partial^k D$ is defined inductively by

$$\partial^k D = \partial(\partial^{k-1} D).$$

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Set $D = \{\, \omega^1 = \cdots = \omega^n = 0 \,\}$ locally. *Cauchy characteristic system* $\mathrm{Ch}(D)$ of D is defined as

$$\mathrm{Ch}(D)(x) := \{\, v \in D(x) \mid v \lrcorner\, d\omega^i \equiv 0 \pmod{\omega^1_x, \ldots, \omega^n_x},\ 1 \leq {}^\forall i \leq n \,\} \quad \text{for } x \in \Sigma$$

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For differential systems D_1 and D_2, $D_1 \cup D_2$ is defined as

$$(D_1 \cup D_2)(x) := D_1(x) + D_2(x) \subset T_x\Sigma \quad \text{for } x \in \Sigma$$

Monge-Ampère system

Monge-Ampère system on a 5-dimensional contact manifold J is an EDS (an ideal)

$$\mathcal{I} = \{\, \theta,\ d\theta,\ \Psi \,\}_{\mathrm{alg}} \subset \Omega^*(J)$$

where θ is the contact form and Ψ is a 2-form that is linearly independent from $d\theta$, modulo θ.

$$\mathcal{I} \text{ is } \begin{bmatrix} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic} \end{bmatrix} \text{ at } u \in J \overset{\mathrm{def}}{\Longleftrightarrow} \mathcal{I}_u \text{ has } \begin{bmatrix} \text{two} \\ \text{one} \\ \text{no} \end{bmatrix} \text{ decomposable 2-covector(s)}$$

Monge-Ampère system

Proposition

Let $\mathcal{I}$ be a hyperbolic Monge-Ampère system around $u \in J$. Then $\mathcal{I}$ has two decomposable 2-forms $\omega^1 \wedge \pi_2, \omega^2 \wedge \pi_2$ around u such that $d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \pmod{\theta}$. Hence

$$\mathcal{I} = \{\, \theta,\, \omega^1 \wedge \pi_1,\, \omega^2 \wedge \pi_2 \,\}_{\mathrm{alg}}.$$

Then *Monge characteristic systems* $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{I}$ is defined by

$$\mathcal{H}_i = \{\, \theta = \omega^i = \pi_i = 0 \,\} \quad \text{for } i = 1, 2.$$

Monge-Ampère system

Proposition

Let $\mathcal{I}$ be a parabolic Monge-Ampère system around $u \in J$. Then $\mathcal{I}$ has one decomposable 2-form $\omega \wedge \pi$ around u such that $d\theta \equiv 0 \pmod{\theta, \omega, \pi}$. Hence

$$\mathcal{I} = \{\, \theta,\ d\theta,\ \omega \wedge \pi \,\}_{\mathrm{alg}}.$$

Then the *Monge characteristic system* $\mathcal{H}$ of $\mathcal{I}$ is defined by

$$\mathcal{H} = \{\, \theta = \omega = \pi = 0 \,\}.$$

prolongation of Monge-Ampère system

Let $\mathcal{I} = \{\, \theta, d\theta, \Psi \,\}_{\mathrm{alg}}$ be a Monge-Ampère system on J. The *prolongation* $(\mathcal{V}_2(\mathcal{I}), D)$ of $\mathcal{I}$ is defined as follows:

$$\mathcal{V}_2(\mathcal{I}) = \bigcup_{u \in J} \mathcal{V}_2(\mathcal{I})_u \xrightarrow{\ \rho\ } J$$

$$\mathcal{V}_2(\mathcal{I})_u = \{v \subset T_u J : \text{2-dim. integral element}\}$$

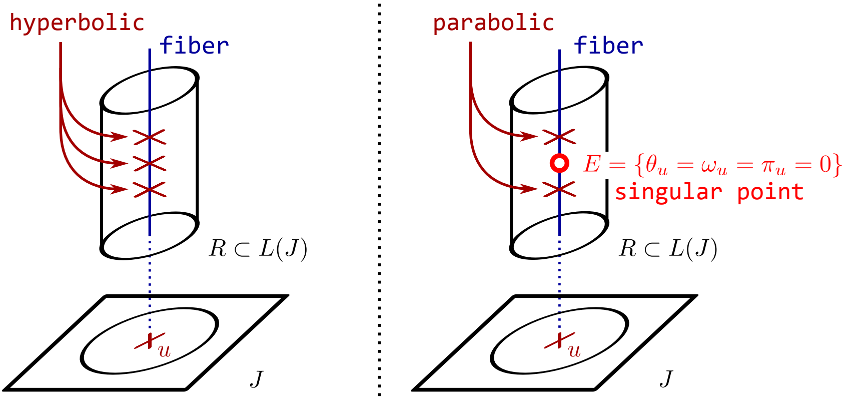
$$\overset{\text{i.e.}}{=} \{v \subset T_u J : \text{2-dim. subspace such that } \theta|_v = d\theta|_v = \Psi|_v = 0\}$$

$$\mathcal{V}_2(\mathcal{I}) = \{v \in L(J) \mid \Psi|_v = 0\} \subset L(J) : \text{subvariety}$$

as $\mathcal{V}_2(\mathcal{I})$ is a submanifold,

$$D(v) := \rho_*^{-1}(v) \quad (v \in \mathcal{V}_2(\mathcal{I}))$$

Let $\mathcal{I}$ be a hyperbolic or parabolic Monge-Ampère system and (R, D) the prolongation of $\mathcal{I}$.



$\mathcal{I} = \{\, \theta,\, \omega^1 \wedge \pi_1,\, \omega^2 \wedge \pi_2 \,\}_{\mathrm{alg}}$ $\qquad$ $\mathcal{I} = \{\, \theta,\, d\theta,\, \omega \wedge \pi \,\}_{\mathrm{alg}}$

hyperbolic: there is no singular points in R. Namely, R is a submanifold of $L(J)$.

parabolic: there is only one singular point E in each fiber $\rho^{-1}(u)$ $(u \in J)$.

$\mathcal{I}$: a hyperbolic Monge-Ampère system
$\mathcal{H}_1, \mathcal{H}_2$: Monge characteristic systems of $\mathcal{I}$
(R, D) : the prolongation of $\mathcal{I}$

Then, for each $i = 1, 2$, there exists a Monge characteristic system $\mathcal{M}_i$ of (R, D) such that

$$\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i).$$

Furthermore, we have the following theorem:

Theorem

$\partial\mathcal{M}_i$ and $\partial^2\mathcal{M}_i$ are (regular) differential systems of rank 3 and 4 respectively and we have

$$\partial^2\mathcal{M}_i \subset \rho_*^{-1}(\partial\mathcal{H}_i), \qquad \partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D) = \rho_*^{-1}(\mathcal{H}_i).$$

$\mathcal{I}$: a parabolic Monge-Ampère system
$\mathcal{H}$: the Monge characteristic system of $\mathcal{I}$
(R, D) : the prolongation of $\mathcal{I}$

Then the Monge characteristic system $\mathcal{M}$ of (R, D) is contained in the pullback of $\mathcal{H}$. Namely,

$$\mathcal{M} \subset \rho_*^{-1}(\mathcal{H}).$$

Furthermore, we have the following theorem:

Theorem

$$\partial\big(\mathcal{M} \cup \mathrm{Ch}(\partial D)\big) = \rho_*^{-1}(\mathcal{H})$$

and $\mathcal{H}$ is completely integrable if and only if $\mathcal{M}$ is completely integrable.
On the other hand, if $\partial\mathcal{M}$ has constant rank and does not coincide with $\mathcal{M}$, then

$$\partial^2\mathcal{M} = \rho_*^{-1}(\mathcal{H}).$$

characterization of hyperbolic Monge-Ampère equations

Theorem

Let (R, D) be a hyperbolic PDE and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of (R, D). If $\partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D)$ drops down to J for each $i = 1, 2$, there exists a Monge-Ampère system $\mathcal{I}$ whose prolongation coincides with (R, D) locally.
Moreover, $\partial\mathcal{M}_1 \cup \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 \cup \mathrm{Ch}(\partial D)$ are then pullbacks of the Monge characteristic systems of the system $\mathcal{I}$.

$$\begin{tikzcd}
L(J) \arrow[d, "\pi"'] & \supset & R \arrow[dll, "\rho = \pi|_R"] \arrow[dl, phantom, "\circlearrowleft"] \\
J & &
\end{tikzcd}$$

$$\begin{tikzcd}
D \supset \partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D) = \rho_*^{-1}(\mathcal{H}_i) \arrow[d, squiggly, "\text{drops down}"] \\
TJ \supset \mathcal{H}_i \quad \text{(corank 3)}
\end{tikzcd}$$

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characterization of hyperbolic Monge-Ampère equations

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Remark

If $\mathrm{Ch}\big(\partial\mathcal{M}_1 \cup \mathrm{Ch}(\partial D)\big) = \mathrm{Ch}(\partial D)$ (resp. $\mathrm{Ch}\big(\partial\mathcal{M}_2 \cup \mathrm{Ch}(\partial D)\big) = \mathrm{Ch}(\partial D)$), $\partial\mathcal{M}_1 \cup \mathrm{Ch}(\partial D)$ (resp. $\partial\mathcal{M}_2 \cup \mathrm{Ch}(\partial D)$) drops down to $J = R/\mathrm{Ch}(\partial D)$. Thus this theorem implies that if one drops down, the other also does.
Hence it is understood that each Monge characteristic system has the complete information whether R is a Monge-Ampère equation or not.

characterization of hyperbolic Monge-Ampère equations

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Example

$$r + s = 0 \quad (\, z_{xx} + z_{xy} = 0 \,)$$ Monge-Ampère equation

$\left(J^2(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^5 : (x, y, z, p, q, r, s, t) \mapsto (x, y, z, p, q) \right)$

$$\mathcal{M}_1 = \{\, \varpi_0 = \varpi_1 = \varpi_2 = dx - dy = ds = 0 \,\}$$
$$\mathcal{M}_2 = \{\, \varpi_0 = \varpi_1 = \varpi_2 = dy = dt + ds = 0 \,\}$$

where $\varpi_0 = dz - pdx - qdy, \varpi_1 = dp + sdx - sdy, \varpi_2 = dq - sdx - tdy$

$$\partial\mathcal{M}_1 = \{\, \varpi_0 = \varpi_1 = dx - dy = ds = 0 \,\}$$
$$\partial\mathcal{M}_2 = \{\, \varpi_0 = \varpi_2 + \varpi_1 = dy = dt + ds = 0 \,\}$$

$$\begin{aligned} \partial\mathcal{M}_1 \cup \mathrm{Ch}(\partial D) &= \{\, \varpi_0 = \varpi_1 = dx - dy = 0 \,\} \\ &= \{\, dz - pdx - qdy = dp = dx - dy = 0 \,\} \\ \partial\mathcal{M}_2 \cup \mathrm{Ch}(\partial D) &= \{\, \varpi_0 = \varpi_2 + \varpi_1 = dy = 0 \,\} \\ &= \{\, dz - pdx - qdy = dp + dq = dy = 0 \,\} \end{aligned}$$

Example

$$r + \tfrac{1}{2}s^2 = 0,\ s \neq 0 \quad (\, z_{xx} + \tfrac{1}{2}(z_{xy})^2 = 0 \,) \qquad \text{non-Monge-Ampère equation}$$

$\left(J^2(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^5 : (x, y, z, p, q, r, s, t) \mapsto (x, y, z, p, q)\right)$

$$\mathcal{M}_1 = \{\, \varpi_0 = \varpi_1 = \varpi_2 = dx - \tfrac{1}{s}dy = ds = 0 \,\}$$
$$\mathcal{M}_2 = \{\, \varpi_0 = \varpi_1 = \varpi_2 = dy = dt + \tfrac{1}{s}ds = 0 \,\}$$

where $\varpi_0 = dz - pdx - qdy$, $\varpi_1 = dp + \frac{1}{2}s^2dx - sdy$, $\varpi_2 = dq - sdx - tdy$

$$\partial\mathcal{M}_1 = \{\, \varpi_0 = \varpi_1 = dx - \tfrac{1}{s}dy = ds = 0 \,\}$$
$$\partial\mathcal{M}_2 = \{\, \varpi_0 = \varpi_2 + \tfrac{1}{s}\varpi_1 = dy = dt + \tfrac{1}{s}ds = 0 \,\}$$

$$\begin{aligned}
\partial\mathcal{M}_1 \cup \mathrm{Ch}(\partial D) &= \{\, \varpi_0 = \varpi_1 = dx - \tfrac{1}{s}dy = 0 \,\} \\
&= \{\, dz - pdx - qdy = dp - \tfrac{1}{2}sdy = dx - \tfrac{1}{s}dy = 0 \,\} \\
\partial\mathcal{M}_2 \cup \mathrm{Ch}(\partial D) &= \{\, \varpi_0 = \varpi_2 + \tfrac{1}{s}\varpi_1 = dy = 0 \,\} \\
&= \{\, dz - pdx - qdy = dq + \tfrac{1}{s}dp - \tfrac{1}{2}dx = dy = 0 \,\}
\end{aligned}$$

characterization of parabolic Monge-Ampère equations

Theorem

Let (R, D) be a parabolic PDE. Let $\mathcal{M}$ denote the Monge characteristic system of (R, D). If $\partial(\mathcal{M} \cup \mathrm{Ch}(\partial D))$ drops down to J, there exists a Monge-Ampère system $\mathcal{I}$ whose prolongation coincides with (R, D) locally.
Moreover, $\partial(\mathcal{M} \cup \mathrm{Ch}(\partial D))$ is then the pullback of the Monge characteristic system of the system $\mathcal{I}$.

$$\begin{tikzcd} L(J) \arrow[d, "\pi"'] & \supset & R \arrow[dll, "\rho = \pi|_R"] \arrow[dl, phantom, "\circlearrowright"] \\ J & & \end{tikzcd}$$

$$D \supset \partial(\mathcal{M} \cup \mathrm{Ch}(\partial D)) = \rho_*^{-1}(\mathcal{H})$$

$$\Big\downarrow \; \texttt{drops down}$$

$$TJ \supset \mathcal{H} \quad \texttt{(corank 3)}$$

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Moreover, $\partial(\mathcal{M} \cup \mathrm{Ch}(\partial D))$ is then the pullback of the Monge characteristic system of the system $\mathcal{I}$.

Theorem

Let (R, D) be a parabolic PDE. Let $\mathcal{M}$ denote the Monge characteristic system of (R, D) and **assume $\partial\mathcal{M}$ has constant rank***. If $\mathrm{Ch}(\partial(\mathcal{M} \cup \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, there exists a Monge-Ampère system $\mathcal{I}$ whose prolongation coincides with (R, D) locally.*

Thank you for your attention

Sketch of proof:

Let us fix a point $v_0 \in R$. Set $D = \{\, \varpi_0 = \varpi_1 = \varpi_2 = 0 \,\}$ around v_0.

structure equation

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2) \end{cases}$$

&

integrability conditions

$$\begin{cases} d^2\varpi_0 \equiv 0 & (\mathrm{mod}\ \varpi_0, \omega^1, \varpi_2) \\ d^2\varpi_0 \equiv 0 & (\mathrm{mod}\ \varpi_0, \omega^2, \varpi_1) \\ d^2\varpi_1 \equiv 0 & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1) \end{cases}$$

$\Longrightarrow$ $^\exists\{\, \varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22} \,\}$: coframe around v_0 such that

$$\begin{aligned} d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} - k_2\, \varpi_2 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2), \\ d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} - h_1\, \varpi_1 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_1), \\ d\omega^1 &\equiv h_1\, \omega^2 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1), \\ d\omega^2 &\equiv k_2\, \omega^1 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2). \end{aligned}$$

where h_1, h_2 are local functions on R.

Therefore we have

$$\partial\mathcal{M}_i = \{\, \varpi_0 = \varpi_i = \omega^i = \widetilde{\pi}_{ii} = 0 \,\},$$
$$\partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D) = \{\, \varpi_0 = \varpi_i = \omega^i = 0 \,\}.$$

If $\partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D)$ drops down to J for each $i = 1, 2$,

1. there exists a differential system $\mathcal{H}_i = \{\, \theta = \widehat{\pi}_i = \widehat{\omega}^i = 0 \,\}$ s.t. $\rho_*^{-1}(\mathcal{H}_i) = \partial\mathcal{M}_i \cup \mathrm{Ch}(\partial D)$ for $i = 1, 2$,
2. and we may have

$$d\theta \equiv (\widehat{K}_1\, \widehat{\omega}^1) \wedge \widehat{\pi}_1 + (\widehat{K}_2\, \widehat{\omega}^2) \wedge \widehat{\pi}_2 \pmod{\theta},$$

   where $\widehat{K}_1$, $\widehat{K}_2$ are functions on J.

3. We construct a Monge-Ampère system

$$\mathcal{I} = \{\, \theta,\, \widehat{\omega}^1 \wedge \widehat{\pi}_1,\, \widehat{\omega}^2 \wedge \widehat{\pi}_2 \,\}_{\mathrm{alg}},$$

   whose Monge characteristic systems are $\mathcal{H}_1$ and $\mathcal{H}_2$.

4. Thus each v around v_0 is an integral element of $\mathcal{I}$, that is, the prolongation of $\mathcal{I}$ coincides with (R, D) locally.