Research on Lagrangian intersections and leaf-wise intersections

Satoshi Ueki

Tohoku University

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The table of contents

Symplectic manifolds

Definition

(P^{2n}, Ω) : symplectic manifold
$:\Leftrightarrow$

(1) Ω is a closed 2-form on P

(2) Ω is nondegenerate

Definition

$L^n \subset P^{2n}$ is Lagrangian
$:\Leftrightarrow \Omega|_L = 0$

Cotangent bundles

For any manifold M, the cotangent bundle T^*M has the natural symplectic form Ω_M:

$$\Omega_M = \sum_{j=1}^{n} dx_j \wedge dy_j$$

(x_j : coordinates of M, y_j : fiber coordinates)

This symplectic form Ω_M is called the canonical form on T^*M

Cotangent bundles

Let ϕ be a 1-form on M.

$$L = \phi(M) : Lag \;\Leftrightarrow\; d\phi = 0 \;\; (\phi \in Z^1(M))$$

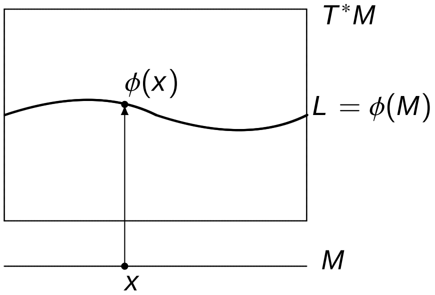

Figure: horizontal Lagrangian submanifold

Lagrangian intersections on cotangent bundles

Let $L_1 = \phi_1(M)$, $L_2 = \phi_2(M)$ $(\phi_1, \phi_2 \in Z^1(M))$ be horizontal Lagrangian submanifolds. Define

$$\Gamma := \phi_2 - \phi_1 \in Z^1(M)$$
$$G := \frac{1}{2}(\phi_1 + \phi_2) : M \to T^*M$$

Then,

$$x \in \mathrm{Zero}(\Gamma) \Rightarrow \phi_1(x) = \phi_2(x)$$
$$\Rightarrow G(x) = \phi_1(x) = \phi_2(x) \in L_1 \cap L_2$$

The intersection of two horizontal Lagrangian submanifolds is given by the zero points of closed 1-form Γ.

A. Weinstein's theorem

A. Weinstein extended this property for Lagrangian intersections of general symplectic manifolds.

Theorem (Weinstein, 1973)

(P, Ω) : symplectic manifold
L_1, L_2 : Lagrangian submanifolds of P
$\Sigma := L_1 \cap L_2$ is a submanifold of P

$\Rightarrow$ If a pair of Lagrangian submanifolds (L_1', L_2') is C^1 close to (L_1, L_2), there are

$$\Gamma \in Z^1(\Sigma)$$
$$G : \Sigma \to P$$

such that

$$p \in \mathrm{Zero}(\Gamma) \Rightarrow G(p) \in L_1' \cap L_2'$$

In particular, consider the case that

- $(P, \Omega) = (T^*M, \Omega_M)$
- $L_1 = L_2 (=: L)$: horizontal Lag

Then, $\Sigma = L \cong M$

If L_1', L_2' are C^1 close to L, then L_1' and L_2' are also horizontal and the intersection is given by the zero points of some $\Gamma \in Z^1(M)$.

Lusternik-Schnirelmann category

Consider the topological invariant $\mathrm{cat}(\Sigma)$, Lusternik-Schnirelmann category of Σ. This is the least number of contractible open sets which cover Σ. Then, the next theorem holds.

Theorem (Lusternik, Schnirelmann)

Σ : a compact manifold
f : a smooth function on Σ
$\Rightarrow$ The number of critical points of f is at least $\mathrm{cat}(\Sigma)$

In the Weinstein's theorem, if Σ is compact and $\Gamma = df$, the number of critical points of f (or zero points of Γ) is at least $\mathrm{cat}(\Sigma)$. Therefore $L'_1 \cap L'_2$ has at least $\mathrm{cat}(\Sigma)$ points.

Coisotropic submanifolds

Definition

$M^{2n-r} \subset P$ is a coisotropic submanifold $(0 \leq r \leq n)$
$:\Leftrightarrow (T_pM)^\Omega \subset T_pM \ \ (p \in M)$

where

$$(T_pM)^\Omega := \left\{ v \in T_pP \;\middle|\; \Omega(v,w) = 0 \ ({}^\forall w \in T_pM) \right\}$$

For example, in the $2n$ dimensional Euclidean space with canonical form $(\mathbb{R}^{2n}, \Omega_0)$, define M as follows

$$M := \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \mid y_{n-r+1} = \cdots = y_n = 0\}$$

Then, M is a coisotropic submanifold.

$$\begin{aligned} M : Lag \ &\Leftrightarrow \ (TM)^\Omega = TM \\ &\Leftrightarrow \ M : coisotropic, \ r = n \end{aligned}$$

Coisotropic submanifolds

For a coisotropic submanifold M, $(TM)^{\Omega} \subset TM$ is a completely integrable distribution in M. This defines a foliation of M.
Let L_p be the leaf through $p \in M$.

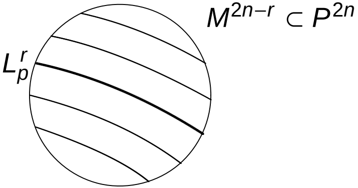

Figure: a coisotropic submanifold with leaves

Since

$$\dim T_pM + \dim(T_pM)^{\Omega} = 2n$$

holds, the dimension of the foliation is r.

Leaf-wise intersections

$M \subset P$: coisotropic submanifold
$\psi \in \mathrm{Symp}(P) := \{\psi \in \mathrm{Diff}(P) \mid \psi^* \Omega = \Omega\}$

Definition

$p \in M$: leaf-wise intersection of ψ
$:\Leftrightarrow \ \psi(p) \in L_p$

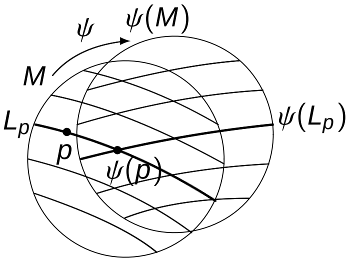

Figure: A leaf-wise intersection

Leaf-wise intersections

$\mathrm{LWI}_M(\psi)$ denotes the set of the leaf-wise intersections of ψ :

$$\mathrm{LWI}_M(\psi) := \{p \in M \mid \psi(p) \in L_p\}$$

Examples of leaf-wise intersections :

- Fixed points of $\psi \in \mathrm{Symp}(P)$ $(r = 0)$
- Lagrangian intersections $(r = n)$
- Periodic orbits in Hamiltonian systems $(r = 1)$

Examples of leaf-wise intersections

(1) ($r = 0$) First example is the case $M = P$. In this case,

$$(T_pM)^\Omega = \{0\},\ L_p = \{p\}$$

Then,

$$\begin{aligned} p \in \mathrm{LWI}_M(\psi) &\Leftrightarrow \psi(p) = p \\ &\Leftrightarrow p \in \mathrm{Fix}(\psi) \end{aligned}$$

In this example, leaf-wise intersections of ψ are fixed points of ψ.

Examples of leaf-wise intersections

(2) ($r = n$) Second example is the case of connected Lagrangian submanifold $M \subset P$.

$$(T_pM)^\Omega = T_pM,\ L_p = M$$

Then,

$$\begin{aligned} p \in \mathrm{LWI}_M(\psi) &\Leftrightarrow \psi(p) \in M \\ &\Leftrightarrow \psi(p) \in M \cap \psi(M) \end{aligned}$$

Since ψ preserves the symplectic form, ψ(M) is also a Lagrangian submanifold of P. In this example, leaf-wise intersections of ψ are Lagrangian intersections of two Lagrangian submanifolds $M,\ \psi(M)$.

Examples of leaf-wise intersections

(3) ($r = 1$) Third example is about periodic orbits of Hamiltonian systems. Let (P, Ω, H) be a Hamiltonian system. Define the Hamiltonian vector field X_H of H by

$$X_H \rfloor \Omega = dH$$

The integral curves of X_H are called orbits of Hamiltonian system (P, Ω, H).

Let $c \in \mathbb{R}$ be a regular value of H and define

$$M := H^{-1}(c).$$

Assume that all orbits in M are periodic.

For example, consider $(P, \Omega) = (\mathbb{R}^{2n}, \Omega_0)$ and the Hamiltonian function

$$H = \frac{\alpha}{2} \sum_{K=1}^{n} \lambda(x_k^2 + y_k^2) \ \ (\alpha > 0,\ \lambda_k \in \mathbb{N})$$

then, all orbits in $M = H^{-1}(c)$ $(c > 0)$ are periodic.

Examples of leaf-wise intersections

Consider the perturbation $H + \tilde{H}$ of this Hamiltonian system. Here,

$$\tilde{H} : P \times \mathbb{R} \to \mathbb{R}, \quad \tilde{H}(\cdot, t) \equiv 0 \ (t \leq a, \ b \leq t)$$

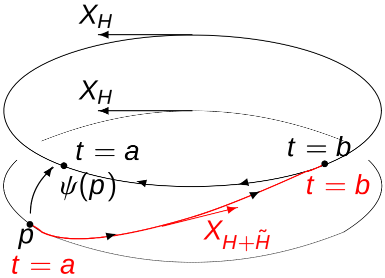

Figure: definition of ψ

Examples of leaf-wise intersections

$M = H^{-1}(c)$ is coisotropic and the leaves of M are orbits of H. If $p \in \mathrm{LWI}_M(\psi)$, $\psi(p)$ lies on the orbit of H through p. Then, the orbit of the perturbed system $(P, \Omega, H + \tilde{H})$ through $p \in \mathrm{LWI}_M(\psi)$ at $t = a$ is periodic.

In this example, the leaf-wise intersections of ψ are the periodic orbits.

J. Moser proved the next theorem about leaf-wise intersections.

Theorem (Moser, 1978)

(P, Ω) : simply connected symplectic manifold
Ω : exact form
$M \subset P$: compact coisotropic submanifold
$\Rightarrow$
If $\psi \in \mathrm{Symp}(P)$ is C^1 close to $id_P : P \to P$, ψ has at least two leaf-wise intersections.

Main theorem

I proved the next theorem by using the Weinstein's theorem.

Main Theorem

(P, Ω) : symplectic manifold
$M \subset P$: coisotropic submanifold
$\Rightarrow$
If $\psi \in \mathrm{Symp}(P)$ is C^1 close to $id_P : P \to P$, there are

$$\Gamma \in Z^1(M)$$
$$G : M \to P \times P$$

such that

$$p \in \mathrm{Zero}(\Gamma) \;\Rightarrow\; \pi_1 \circ G(p) \in \mathrm{LWI}_M(\psi)$$

Particularly, if M is compact and $\Gamma = df$, then ψ has at least $\mathrm{cat}(\Sigma)$ leaf-wise intersections.

Remarks on main theorem

In this theorem, suppose that P is simply connected. Then Γ is exact. Moreover, suppose that M is compact. Then any smooth function on M has at least two critical points. Therefore, ψ has at least two leaf-wise intersrctions.

Hence,

$$\text{Main Theorem} \Rightarrow \text{Theorem (Moser)}$$

Remarks on main theorem

There is a LS category for foliated manifolds. We only have the evaluation of the critical points of a function which is constant on each leaves by this LS category.

On the other hand, the function f on main theorem is not necessarily constant on each leaves.

The extended LS category is useless for evaluating the number of leaf-wise intersections of a symplectomorphism ψ.

Thank you very much!